An Essay on the Calculus of Enlargement.

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A. Outline.

1. The Calculus of Enlargement is, from one point of view, an extension of the Calculus of Finite Differences; from another, an extension of the Calculus of Operations. It comprises, as its most important branch, the Differential Calculus, included in which is the Calculus of Variations. The scope of this new science is, therefore, comprehensive. Its method, on the other hand, is simple. My present object is not to exhibit it in a methodical treatise, but merely to give a preliminary sketch of it, and so to publish its discovery.

2. The Calculus of Enlargement is, from one point of view, an extension of the Calculus of Finite Differences. It has for its basis the well-known operation $\mathrm{E} = 1 + \Delta$, or rather, as I prefer to state it, the operation E^h, where

$$\mathrm{E}^h x = x + h, \tag{1}$$

$$\mathrm{E}^h \phi(x) = \phi(x + h). \tag{2}$$

I call this operation Enlargement.*

*The term is elliptical, since by the Enlargement of a function is meant that change which results from the enlargement of the variable. It would probably be hard, however, to find a more appropriate name. The word Enlargement has this further advantage, that its initial letter has been long in use as the symbol of this operation. It will, of course, sometimes be necessary to call that a negative enlargement which is in reality a diminution, just as the word increment sometimes denotes that which is arithmetically a decrement.

3. From another point of view, the Calculus of Enlargement is a modification and extension of the Calculus of Operations, or doctrine of the Separation of Symbols. The symbolic method, as usually explained, concerns itself with the symbol of differentiation, $\frac{d}{dx}$ or D, and with the various functions of that symbol, considered apart from the subject of operation; and among these functions of D is $\text{E} = \varepsilon^{\text{D}}$. The Calculus of Enlargement, on the other hand, regards E as the fundamental symbol, and takes cognizance of other symbols only in case they are, and because they are, functions* of E. Among such, of course, is $\text{D} = \log \text{E}$. If we conceive the symbolic method to be modified and defined in this manner, and to be ranked as a science by itself, instead of a mere auxiliary principle; and if we further conceive this science to be so extended as to include not only, as at present, the separate treatment of symbols of operation, but also a complete discussion of the operations denoted by such symbols, their definitions, uses and consequences, we shall have in mind the Calculus of Enlargement.†

4. The theory of differentiation, comprising the Differential and Integral Calculus and their applications, and including the Calculus of Variations, of which the fundamental operation is differentiation with respect to an imagined variable, forms the most important branch of the Calculus of Enlargement. The algebra of the functions of E is subject to all the laws of ordinary algebra; and *the theory of differentiation is that part of the calculus which corresponds to the theory of logarithms in algebra.*

5. In this manner is effected the orderly unification of those branches of science which I have mentioned. Writers on finite differences have said repeatedly that a differential is but a certain kind of difference, so that the differential calculus may be regarded as a part of the former science; but the connection thus indicated is so trivial, and its consequences are so insignificant, that the claim excites no attention. Nevertheless, it will be agreed that the boundary line between these two branches is but indistinct, and that their formal union, supposing it to be accomplished in a natural and simple manner, is a result to be desired. The obvious connecting link is the equation

*By "function" of x, throughout this essay, I mean a quantity which can be expressed by a series of terms, each of the form $\text{A}x^{a}$, where A and a are independent of x, and are not necessarily integral or positive.

†"This branch of science [the Calculus of Operations] is yet in its infancy, but already it has been the instrument of greatly extending the domains of science, and we may reasonably look to it for the next great step in the direction of mathematical progress."—DAVIES & PECK, *Mathematical Dictionary*, p. 401.

$\text{E} = \varepsilon^{\text{D}}$, or its converse, $\text{D} = \log \text{E}$. The union must be effected, if at all, in one of two ways. On the one hand, we may begin by defining $\frac{d}{dx}$ or D, and then proceed to $\text{E} = \varepsilon^{\text{D}}$ and $\Delta = \varepsilon^{\text{D}} - 1$. This is the unnatural order hitherto tacitly followed, not only by those writers who have appended a chapter or two on finite differences to their treatises on the differential calculus, but also in works devoted to finite differences, all of which, in late years, assume a prior knowledge of differentiation. A student is first taught differentiation; later, he learns the doctrine of the separation of symbols, and finally, if sufficiently zealous, he takes up finite differences. In the latest book on this subject, that of Boole, the reader is referred, for the readiest proof that D, E, and Δ are mutually subject to algebraic discussion, to a passage in that author's work on differential equations. We may, on the other hand, adopt the more natural order, defining E first, and giving afterwards, as one of its functions,

$$\text{D} = \log \text{E}. \tag{3}$$

This well-known equation has not hitherto, I believe, been proposed as the definition of the symbol, and therefore of the operation, of differentiation. To say that Differentiation is the logarithm of Enlargement would seem, and possibly be, a quasi-metaphysical absurdity; but we can and should say that Differentiation is that operation whose symbol is the logarithm of the symbol of Enlargement. Of the two operations, the simpler should be defined the earlier. Now

$$\text{E}\phi(x) = \phi(x+1) \tag{4}$$

is a simpler statement than

$$\text{D}\phi(x) = \frac{\varphi(x+h) - \varphi(x)}{h}_{[h=0]}. \tag{5}$$

These operations, E and D, are functions of each other, and whichever is defined last must be expressed in terms of the other. That D shall be defined in terms of E is the most important feature of the Calculus of Enlargement.

6. The theory of differentiation, I have said, is that part of the calculus which corresponds to the theory of logarithms in algebra. This proposition leads directly to very important consequences. Since D is a function of E, all theorems which may be discovered concerning $\phi(\text{E})$ will be true of D, and also, more generally, true of $\psi(\text{D})$, supposing $\phi(x) = \psi(\log x)$. I shall show that in this manner the known theorems of the differential calculus can be proved, and novel truths discovered, by a method almost startling from its simplicity. Again, from every known or ascertainable proposition in the

theory of logarithms we shall derive at once a corresponding proposition in the theory of differentiation; while, conversely, additions will be made to the theory of logarithms analogous to known truths in that of differentiation. Finally, from every known or ascertainable equation representing $\log x$ in terms of x, or of any simple function of x, we shall derive a corresponding explanation, or practical definition, of the operation of differentiation; including not only the well-known explanation conveyed by (5), but also others in unlimited number, some of them very serviceable.

7. An outline of the Calculus of Enlargement has now been presented. Its brevity places it under a certain disadvantage, yet to treat the subject properly would require the preparation of a complete digest of the Calculus. Not having immediate opportunity to elaborate a work covering so much ground, I am compelled to confine myself for the present to a statement of the general principles on which such a digest should be prepared. The remainder of this essay will be devoted to the presentation of such new special theories as seem needed to complete the system.

B. Suggestions in Detail.

I. *Theory of Logarithms.*

8. An obvious objection to the use of $\log \mathrm{E}$ as the definition of D lies in the obscurity of the idea of the logarithm of an operative symbol; and to go further back, this obscurity is due to the difficulty of comprehending logarithms at all. It is said by De Morgan (*Calculus*, p. 126) that the only definition of $\log x$ used in analysis is y, where $\varepsilon^y = x$. When x and y are not numerical quantities, this is clearly unintelligible. It is certainly impossible to understand the expression ε^{D}, so frequently employed, if we suppose it to mean, as it must mean unless otherwise defined, the Dth power of the constant ε. Even when x and y are numbers, the definition is but indirect at the best. The alternative definitions which I have to suggest correspond identically with the explanations which will, further on, be given concerning $\mathrm{D} = \log \mathrm{E}$. We may consider $\log x$ to be y, where $x = 1 + y + \frac{1}{2} y^2 + \frac{1}{2.3} y^3 + \ldots$; or we may regard it as a vanishing fraction, or as an infinite series. The simplest series, and probably the most intelligible definition, is Mercator's well-known series,

$$\log (1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \ldots. \tag{6}$$

9. Whatever definition of log x be adopted, it will be desirable to lay down the following definition of an antilogarithm. The series $1 + y + \frac{1}{2} y^2 + \frac{1}{2.3} y^3 + \ldots$ is a function of y; let it be known as the antilogarithm of y, and let it be denoted by the functional symbol ε^y. We may proceed as follows to investigate the properties of this symbol. By actual multiplication of the series, we shall find that $\varepsilon^x \varepsilon^y = \varepsilon^{x+y}$, where x and y may have any possible meaning. By an obvious extension of the same principle,

$$(\varepsilon^x)^h = \varepsilon^{xh}, \tag{7}$$

h being any numerical quantity, positive or negative. Putting $x = 1$, we see that $(\varepsilon^1)^h = \varepsilon^h$, from which we see that ε^h is equal to a certain constant raised to a power denoted by h. It is usual to call this constant ε. When h is not a symbol of quantity, it will be safe to regard ε^h as a symbol merely, according to its definition. In short, for all meanings of x, we have the well-known exponential theorem,

$$\varepsilon^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{2.3} x^3 + \ldots, \tag{8}$$

where, if x is a symbol of quantity, ε is a constant, whose value may be found by putting $x = 1$:

$$\varepsilon = 1 + 1 + \frac{1}{2} + \frac{1}{2.3} + \ldots. \tag{9}$$

Having established this understanding concerning the symbol ε, we may define log x to be y, where $x = \varepsilon^y$, or where $x = 1 + y + \frac{1}{2} y^2 + \ldots$; and the various theorems concerning logarithms may be developed in the usual manner.

10. Another and, when duly weighed, most satisfactory definition may be derived from any one of an unlimited number of vanishing fractions, special cases of the general form

$$\log x = \frac{x^{(1-a)h} - x^{-ah}}{h}, \tag{10}$$

where h is infinitely reduced, that is to say, more briefly, where $h = 0$. This fraction is doubtless novel, though one case of it, where $a = 0$, is known. Even that case has not, I presume, been suggested heretofore as a definition. From (10) we have at once, substituting the equivalent series for ε^y,

$$\log \varepsilon^y = y. \tag{11}$$

The various theorems pertaining to logarithms may be derived with the

utmost facility by the aid of these vanishing-fraction definitions. Thus, if $a = 0$, we have, by expansion,

$$\log(\varepsilon^x \varepsilon^y) = \frac{(\varepsilon^x \varepsilon^y)^h - 1}{h}_{[h=0]} = x + y = \log \varepsilon^{x+y}, \tag{12}$$

$$\log(1+x) = \frac{(1+x)^h - 1}{h}_{[h=0]} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots. \tag{13}$$

11. Equation (6) furnishes, perhaps, the most intelligible definition of a logarithm. It is easy to form the idea of a function of the form $x - \frac{1}{2}x^2$, and the conception is not rendered more difficult by adding a term $\frac{1}{3}x^3$, or a multitude of terms similar in form. The notion of the sum of a series of integral powers is simpler than that of a vanishing fraction, and is also simpler than the customary notion of a logarithm, which involves, in an obscure and inverted manner, a fractional, or rather incommensurable, power of a strange looking constant. For instance,

$$\log \frac{3}{2} = \frac{1}{2} - \frac{1}{2.2^2} + \frac{1}{3.2^3} - \ldots = 0.405 + \tag{14}$$

is a more intelligible definition than $\log \frac{3}{2} = y$, where $\frac{3}{2} = \varepsilon^y$, where $\varepsilon = 1 + 1 + \frac{1}{2} + \ldots$. When x lies between 1 and -1, the series (6) is convergent, and the value of the logarithm may be obtained by approximation. When x is algebraically greater than 1, the series is divergent, but it may readily be shown that its sum is finite. Assuming what will shortly be proved, that if $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$, $x = y + \frac{1}{2}y^2 + \frac{1}{2.3}y^3 + \ldots$, one series being algebraically the reverse of the other, we observe that the latter series is essentially convergent, and that when $y = 0$, $x = 0$; when $y = \infty$, $x = \infty$; and when y varies continuously from 0 to ∞, x does the same, having a positive finite value for every positive finite value of y. The converse proposition is, therefore, true, that y, or $\log(1+x)$, is positive and finite for every positive finite value of x. The assumption just made is legitimate, for the proof of the reversion will certainly be accepted when $x < 1$, and the law of the coefficients of the reverse series cannot be different when x has any other value.

12. The various theorems relating to logarithms may easily be derived from this definition. Thus, by the binomial theorem, supposing $1 < a < 2$,

$$a^x = (1 + a - 1)^x = 1 + x(a-1) + x\frac{x-1}{2}(a-1)^2 + \ldots, \tag{15}$$

which may be written

$$a^x = 1 + xc_1 + x^2c_2 + \ldots, \tag{16}$$

where $c_1 = \log a$. Hence,

$$a^{x+y} = a^y(1 + xc_1 + x^2c_2 + \ldots) = 1 + (x+y)\,c_1 + (x+y)^2\,c_2 + \ldots. \tag{17}$$

Placing the coefficients of x equal to each other, a proceeding to which in this case no objection can be urged, we have

$$a^y c_1 = c_1 + 2c_2y + 3c_3y^2 + \ldots. \tag{18}$$

But

$$a^y c_1 = c_1 + c_1c_1y + c_1c_2y^2 + \ldots; \tag{19}$$

hence $c_2 = \frac{1}{2}\,c_1^2$, $c_3 = \frac{1}{3}\,c_1c_2 = \frac{1}{2\,.\,3}\,c_1^3$, and so on, so that

$$a^y = 1 + y\log a + \frac{1}{2}\,y^2(\log a)^2 + \frac{1}{2\,.\,3}\,y^3(\log a)^3 + \ldots. \tag{20}$$

Since, by (7),

$$\left(\varepsilon^{\log a}\right)^h = \varepsilon^{h\log a}, \tag{21}$$

we perceive, on comparison of (20) with (8), that, if $n = \log a$,

$$\varepsilon^{\log a} = a, \tag{22}$$

$$\varepsilon^{ny} = 1 + ny + \frac{1}{2}\,n^2y^2 + \ldots. \tag{23}$$

The applicability of (22) is limited by the supposition that $1 < a < 2$. This limitation may now be removed. Suppose $m = a - 1$, then $n = \log(1 + m)$, and if from the series $\log(1 + m)$ we seek by reversion to determine the value of m, we find it to be, however far the reversion may be carried,

$$m = n + \frac{1}{2}\,n^2 + \frac{1}{2\,.\,3}\,n^3 + \ldots. \tag{24}$$

We see by (22) that the law of this series is true for certain values of n, and the coefficients, independent of n, must be the same for all other values, so that (22) is universally true. Hence, for all meanings of x and y,

$$\varepsilon^{\log(xy)} = xy = \varepsilon^{\log x}\varepsilon^{\log y} = \varepsilon^{\log x + \log y}, \tag{25}$$

$$\log(xy) = \log x + \log y; \tag{26}$$

and again, from (21), h being a symbol of quantity, and u having any assignable meaning,

$$u^h = \varepsilon^{h\log u}, \tag{27}$$

$$\log u^h = h\log u. \tag{28}$$

13. That $\log x$ may be expressed in terms of x is well known. It is only necessary to write out the development of

$$\log x = \log\frac{1+x}{1+x^{-1}} = \log(1+x) - \log(1+x^{-1}). \tag{29}$$

It is possible that the fact has not been noticed that an unlimited number of similar developments may be produced, the general form being

$$\log x = \frac{1}{n}\,[\log(1+x^n) - \log(1+x^{-n})], \tag{30}$$

n having any value, positive or negative.

14. To explain the meaning of $\mathrm{D} = \log \mathrm{E}$, we must employ such expressions as can be found equivalent to $\log x$, substituting E for x; and it is desirable, to ensure breadth of view, to find as many such expressions as possible. I shall now present a *general logarithmic series*, which will be found to include as special cases not only two or three expressions already known, but also several important expressions hitherto unknown, besides an unlimited number of less useful variations. Let $y = x^{h(1-a)} - x^{-ha}$;* then

$$\log x = \frac{y}{h}\left(1 + \frac{2a-1}{2}\,y + \frac{3a-1}{2}\,\frac{3a-2}{3}\,y^2 + \ldots\right). \tag{31}$$

This series may be derived from (6) by writing, for $(1+x)$, $x^h = 1 + yx^{ha}$, and performing the necessary successive substitutions; but this process does not seem capable of furnishing a satisfactory algebraic demonstration. For the present, I must content myself with saying that the law of the series may be verified by reversion to any given extent, and that it may be demonstrated at once by Lagrange's theorem, as well as by another, and perhaps simpler, expansion theorem which will be given further on. The more important special cases are separately susceptible of algebraic proof, so that the temporary lack of a complete demonstration of the general series is not perceptibly detrimental, though certainly to be regretted.

15. Since a and h may have any value, the number of logarithmic formulæ which may be deduced from the general series is infinite. For h, however, but two values, 1 and 0, can advantageously be taken, all other values giving results substantially equivalent to those obtained when $h = 1$. Let us first consider the case where $h = 0$, and consequently $y = 0$. In this case all terms vanish except the first, which we may call the *general logarithmic vanishing fraction:*

$$\log x = \frac{y}{h} = \frac{x^{(1-a)0} - x^{-a0}}{0} = \frac{0}{0}. \tag{32}$$

We interpret this, of course, to mean that $\log x$ is the limit of the ratio of $x^{(1-a)h} - x^{-ah}$ and h, when h is indefinitely reduced. I shall have frequent

*Formulæ more symmetrical, though less simple, may be obtained by writing $\frac{1}{2}(1-b)$ for a.

occasion to use the symbol 0 to express a variable to which the value 0 is to be assigned, as in the present instance. Concerning vanishing fractions in general I shall have more to say later. A simple proof of (32) may be had by expanding, in terms of h, the ratio mentioned, employing the exponential theorem, and afterwards making $h = 0$. In fact, a formula still more general in form may thus be obtained. For, u being any function of x having a finite logarithm,

$$u^h x^h = 1 + h\,(\log u + \log x) + h^2\mathrm{P}\,,\ suppose\,; \tag{33}$$

$$u^h = 1 + h\log u + h^2\mathrm{Q}\,. \tag{34}$$

Subtracting, dividing by h, and making $h = 0$, we have the general formula in question, which, like (32), is probably new,

$$\log x = \frac{u^0 x^0 - u^0}{0}\,. \tag{35}$$

If, in (32), we put $a = 0$, $a = 1$, $a = \frac{1}{2}$, respectively, we have

$$\log x = \frac{x^0 - 1}{0}\,, \tag{36}$$

$$\log x = \frac{1 - x^{-0}}{0}\,, \tag{37}$$

$$\log x = \frac{x^{\frac{0}{2}} - x^{-\frac{0}{2}}}{0}\,, \tag{38}$$

of which equations the first is known.

16. Before making $h = 0$, let $a = -\frac{c}{h}$, where c is any arbitrary quantity, either positive or negative, so that $\frac{y}{h} = x^c\,\frac{x^h - 1}{h} = x^c \log x$. Let $z = -ay = \frac{cy}{h} = cx^c \log x$. Then, from (31),

$$\log x = \frac{1}{c}\left(z - z^2 + \frac{3}{2}\,z^3 - \frac{4^2}{2\,.\,3}\,z^4 + \frac{5^3}{2\,.\,3\,.\,4}\,z^5 - \ldots\right). \tag{39}$$

We may notice particularly two special cases. If $c = 1$,

$$\log x = x\log x - (x\log x)^2 + \frac{3}{2}\,(x\log x)^3 - \ldots\,; \tag{40}$$

while if $c = -1$,

$$\log x = \frac{\log x}{x} + \left(\frac{\log x}{x}\right)^2 + \frac{3}{2}\left(\frac{\log x}{x}\right)^3 + \ldots\,. \tag{41}$$

These interesting series appear to be new. The first of the three is not in reality more general than the others, since it may be derived from the second

by writing x^c, and from the third by writing x^{-c}, for x. We may verify (40) to any desired extent by reversion of

$$x \log x = \varepsilon^{\log x} \log x = \log x + (\log x)^2 + \frac{1}{2} (\log x)^3 + \dots . \tag{42}$$

17. If $h = 1$, we have the general series in the following simplified form, still probably novel,

$$\log x = y + \frac{2a-1}{2} y^2 + \frac{3a-1}{2} \frac{3a-2}{3} y^3 + \dots , \tag{43}$$

where $y = x^{1-a} - x^{-a}$. If $a = 0$, $y = x - 1$, and

$$\log x = (x-1) - \frac{1}{2} (x-1)^2 + \dots , \tag{44}$$

as by (6). If $a = 1$, $y = 1 - x^{-1}$, and

$$\log x = (1 - x^{-1}) + \frac{1}{2} (1 - x^{-1})^2 + \dots , \tag{45}$$

which expression, due, I believe, to Lagrange, may be regarded as conjugate to the one preceding. If $a = \frac{m}{n}$, a proper fraction, the coefficients of y^n, y^{2n}, &c., disappear.

18. If $h = 1$ and $a = \frac{1}{2}$, the resulting series is remarkable, since every alternate term disappears, and those terms which remain converge rapidly when x is not far from 1. Supposing $t = \frac{y}{2} = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{2}$, the series is as follows:

$$\log x = 2 \left(t - \frac{1}{2} \frac{t^3}{3} + \frac{1}{2} \frac{3}{4} \frac{t^5}{5} - \frac{1}{2} \frac{3}{4} \frac{5}{6} \frac{t^7}{7} + \dots \right) .^* \tag{46}$$

The law of the coefficients may be proved as follows. Let $u = \frac{x-1}{x+1}$; then $x = \frac{1+u}{1-u}$, and $\log x = \log (1+u) - \log (1-u)$. In the expansion of this expression let u be replaced by its equivalent $t (1+t^2)^{-\frac{1}{2}}$, and let the several powers of the binomial $1 + t^2$ be developed. It will be found that the coefficient of t^n, for even values of n, is 0; for odd values, let $m = \frac{1}{2} n$, and the coefficient of t^n will be composed of $m + \frac{1}{2}$ terms of the series

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*A special case of this formula, giving $\log \mathrm{E}$ in terms of $\frac{\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}}{2}$, has for some years been known, and it is surprising that its generalized application to all logarithms should not heretofore have been suggested. The formula for $\log \mathrm{E}$ was first published, so far as I am aware, in a communication made in 1865 by Hansen to the Royal Society of Saxony; but he did not assign the law of the series, which was communicated by Mr. T. B. Sprague in 1871 to Mr. W. M. Makeham, and published in the *Journal of the Institute of Actuaries.* Mr. Sprague's proof was by the method of indeterminate coefficients, with differentiation.

$\frac{1}{m}\left(1-m+m\frac{m-1}{2}-m\frac{m-1}{2}\frac{m-2}{3}+\ldots\right)$, the sum of which, by a known algebraic formula, is $\frac{(1-m)^{(m-\frac{1}{2}}}{m\,.\,1^{(m-\frac{1}{2}}}$, where $x^{(r}=x(x+1)(x+2)\ldots(x+r-1)$.

Thus, if $n=1$, the coefficient is $\frac{\left(\frac{1}{2}\right)^{(0}}{\frac{1}{2}\,.\,1^{(0}}=2$; if $n=3$, it is $\frac{\left(-\frac{1}{2}\right)^{(1}}{\frac{3}{2}\,.\,1^{(1}}=-\frac{1}{3}$ or $2\left(-\frac{1}{2}\frac{1}{3}\right)$, and so on, as in (46).

19. For substitution in (46), let $z=\frac{1}{x-1}$, so that $x=\frac{z+1}{z}$; also, let $u=4z\,(z+1)=t^{-2}$. Making these substitutions, and multiplying both members by $\surd(z\,[z+1])$, we have

$$\surd(z\,[z+1])\log\frac{z+1}{z}=1-\frac{1}{2}\frac{1}{3u}+\frac{1}{2}\frac{3}{4}\frac{1}{5u^2}-\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{7u^3}+\ldots, \tag{47}$$

a formula which will be found useful in the computation of logarithms, and which may be compared with the known series,

$$\left(z+\frac{1}{2}\right)\log\frac{z+1}{z}=1+\frac{1}{3v}+\frac{1}{5v^2}+\ldots, \tag{48}$$

where $v=4\left(z+\frac{1}{2}\right)^2$. In the one, we have, for determining $\log\frac{z+1}{z}$, to make use of $\surd(z\,[z+1])$, the geometrical mean between z and $z+1$, while in the other we have to employ $z+\frac{1}{2}$, the arithmetical mean. Suppose that log 3, and therefore log 9, are known, and that it is desired to calculate log 10. Employing the usual formula (48), we have a very convergent series,

$$\frac{19}{2}\log\frac{10}{9}=1+\frac{1}{1083}+\frac{1}{651605}+\ldots; \tag{49}$$

but by (47) we obtain a series still more highly convergent,

$$\surd 90\log\frac{10}{9}=1-\frac{1}{2160}+\frac{1}{1728000}-\ldots. \tag{50}$$

20. I conclude these suggestions concerning the theory of logarithms by presenting two novel approximative expressions. First,

$$\log x=\frac{x+1}{3}\,\frac{x^3-1}{x^3+x},\ \textit{nearly}, \tag{51}$$

whenever x is not far from 1. By development in terms of $x-1$, we find that this expression differs from $\log x$ by a quantity arithmetically less than

$\frac{(x-1)^5}{20}$. For example, if $x = 0.99$, $\log x = -0.010050335858$, nearly, a result too great by 4 in the 12th place of decimals. Again,

$$\log x = 2\,\frac{x-1}{3}\left(\frac{1}{x+1} + \sqrt{\frac{1}{x}}\right), \textit{ nearly}, \tag{52}$$

whenever x is not far from 1, the error being arithmetically less than $\frac{(x-1)^5}{320}$. For example, $\log 0.99 = -\frac{2}{597} - \frac{\sqrt{11}}{495}$, nearly, which is correct in the 12th place.

II. *General Theory of Operations.*

21. Algebra takes any symbols subject to these three laws,

$$x(y+z) = xy + xz, \tag{53}$$

the law of distribution;

$$xy = yx, \tag{54}$$

the law of commutation, and

$$x^m x^n = x^{m+n}, \tag{55}$$

the law of indices, and proves that certain theorems concerning such symbols follow necessarily from the laws. The various theorems of algebra are as true of all operative symbols subject to the three laws in question as they are of common symbols of quantity. Any correct process of reasoning applied to such symbols of operation produces correct results, by precisely that kind of proof which it is necessary to employ regarding symbols of quantity. There is no novelty in these preliminary statements. At first, the symbolic method was used as an instrument of discovery only with the utmost caution, and its results were not fully accepted until otherwise verified. Its absolute trustworthiness has, however, been established by the clearest methods of demonstration, and no mathematician now doubts the algebraic truth of any intelligible symbolic result. If any doubt remains, it is when a divergent series appears.

22. I would define a simple operation to be one which changes a function by alteration of the variable. For example, the change of ϕx (it would be more formal to write $\phi[x]$, but I shall omit the brackets where no ambiguity can arise) into $\phi\psi x$ is a simple operation. All simple operations are obviously

distributive, though some distributive operations are not simple. For example, $\mathrm{D}x^m = mx^{m-1}$ is not a simple operation.

23. Let s^h represent any operation such that

$$\mathrm{s}^h\phi x = \phi\psi^{-1}(\psi x + h)\,. \tag{56}$$

Here s indicates the kind of operation, depending on the form of ψ, and h represents the degree to which it is carried.

$$\begin{aligned}\mathrm{s}^m\mathrm{s}^n\phi x = \mathrm{s}^m\phi\psi^{-1}(\psi x + n) &= \phi\psi^{-1}(\psi\psi^{-1}[\psi x + n] + m) \\ = \phi\psi^{-1}(\psi x + m + n) &= \mathrm{s}^{m+n}\phi x\,.\end{aligned} \tag{57}$$

The operation s^h is, therefore, subject to the law of indices, and it will similarly be seen that s^m and s^n are commutative with each other and with constants, that is to say,

$$\mathrm{s}^m\mathrm{s}^n c\phi x = c\mathrm{s}^n\mathrm{s}^m\phi x\,. \tag{58}$$

For s^1 it will be sufficient to write s, without the index. The index h may, of course, have any value, whole or fractional, positive or negative, or it may even be a meaningless symbol; meaningless, that is, until some meaning is arbitrarily assigned to it.

24. Let $f_1\mathrm{s}$ be any function of s, the general form being

$$f_1\mathrm{s} = a_1\mathrm{s}^{p_1} + a_2\mathrm{s}^{p_2} + a_3\mathrm{s}^{p_3} + \ldots\,, \tag{59}$$

where $a_1, a_2 \ldots, p_1, p_2 \ldots$, are independent of s, and have any assignable meaning, so that

$$f_1\mathrm{s}\phi x = (a_1\mathrm{s}^{p_1} + \ldots)\,\phi x = a_1\phi\psi^{-1}(\psi x + p_1) + \ldots\,. \tag{60}$$

It will be seen, on examination, that all such functions are distributive and repetitive, and it is easy to show that they are also commutative. Let $f_2\mathrm{s}$ be another such function, say

$$f_2\mathrm{s} = b_1\mathrm{s}^{q_1} + b_2\mathrm{s}^{q_2} + \ldots\,; \tag{61}$$

then

$$f_1\mathrm{s}f_2\mathrm{s}\phi x = f_2\mathrm{s}f_1\mathrm{s}\phi x\,. \tag{62}$$

The general term of $f_1\mathrm{s}$ is, let us say, $a_m\mathrm{s}^{p_m}$, and that of $f_2\mathrm{s}$, $b_n\mathrm{s}^{q_n}$; then the general term of $f_1\mathrm{s}f_2\mathrm{s}$ will be $a_m\mathrm{s}^{p_m}b_n\mathrm{s}^{q_n}$, and that of $f_2\mathrm{s}f_1\mathrm{s}$ will be $b_n\mathrm{s}^{q_n}a_m\mathrm{s}^{p_m}$, which, by (58), are seen to be equivalent expressions. It follows that all terms of the two expansions correspond, so that the operations denoted by $f_1\mathrm{s}$ and $f_2\mathrm{s}$, that is to say, all functions of s, including constants, are commutative with each other. It follows that all functions of s may be combined or transformed in any usual algebraic manner, apart from the subject upon which they operate.

25. The consideration of the functions of S, after the form of ψ, and therefore that of S, have been assigned, constitutes, in the nomenclature of this essay, a Calculus. Since ψ, and therefore S, may have any form, there may be an infinite number of such branches of science called Calculus. The operations comprised under one calculus will not usually be commutative with those of another, but two or more operations belonging to different systems may be treated separately from the subject on which they are performed, provided care be taken not to change their order.

26. In every calculus the most important branch is that which corresponds to the theory of logarithms in algebra. There are several important theorems which are thus, in a sense, common to all such systems, having their common origin in the theory of logarithms. Whatever be the meaning of S, let $\mathrm{R} = \log \mathrm{S}$; then from paragraphs 13–18 we shall derive at once a number of expressions giving R in terms of S or of simple functions of S, expressions which it is not necessary, for present purposes, to write out. As an illustration, we have from (36)

$$\mathrm{R} = \frac{\mathrm{S}^0 - 1}{0}. \tag{63}$$

Let $\psi x = x^{-1}$, and let what S becomes under this supposition be denoted by H; then

$$\mathrm{H}^h \phi x = \phi \frac{x}{1 + xh}, \tag{64}$$

and the calculus composed of all functions of the symbol H may be called the Calculus of H. To show the use of (63), let $\phi x = x^n$; then, if $\mathrm{G} = \log \mathrm{H}$,

$$\mathrm{G} x^n = \frac{x^n (1 + xh)^{-n} - x^n}{h}, \tag{65}$$

where $h = 0$, whence after development, assuming the binomial theorem,

$$\mathrm{G} x^n = - n x^{n+1}. \tag{66}$$

Again,

$$\mathrm{G} \log x = \frac{\log x - \log (1 + xh) - \log x}{h}, \tag{67}$$

where $h = 0$, whence

$$\mathrm{G} \log x = - x. \tag{68}$$

27. The widest generalization of Taylor's theorem which I have been able to discover is that which gives S^h in terms of $h\mathrm{R}$. Since $\mathrm{S}^h = \varepsilon^{h\mathrm{R}}$, we have from (23)

$$\mathrm{S}^h = 1 + h\mathrm{R} + \frac{1}{2} h^2 \mathrm{R}^2 + \frac{1}{2.3} h^3 \mathrm{R}^3 + \dots . \tag{69}$$

As a single illustration of this theorem, let us, in the Calculus of H as before, expand $\text{H}^h \log x = \log x - \log (1 + xh)$:

$$\log x - \log (1 + xh) = \log x + h\text{G} \log x + \frac{1}{2} h^2\text{G}^2 \log x + \ldots$$

$$= \log x - hx + \frac{1}{2} h^2x^2 - \frac{1}{3} n\, x^3 + \ldots. \qquad (70)$$

28. A connecting link between any calculus, say that of S, and any other, say that of S′, may be found as follows. From (56),

$$\text{S}^n\phi x = \phi\psi'^{-1}(\psi'x + \psi'\psi^{-1}[\psi x + n] - \psi'x) = \text{S}'^{(\text{S}^n - 1)\psi'x}\phi x. \qquad (71)$$

From (69), writing t and t' for series containing n^2 as a factor, we derive this transformation of (71),

$$(1 + n\text{R} + t)\,\phi x = (1 + [\text{S}^n - 1]\,\psi'x\,.\,\text{R}' + t')\,\phi x = (1 + [n\text{R} + t]\,\psi'x\,.\,\text{R}' + t')\,\phi x; \qquad (72)$$

whence, equating the coefficients of n,

$$\text{R}\phi x = \text{R}\psi'x\,.\,\text{R}'\phi x. \qquad (73)$$

Let $\phi x = \psi x = \psi' x$; then $\text{R}' = \text{R}$, and

$$\text{R}\psi x = \text{R}\psi x\,.\,\text{R}\psi x, \qquad (74)$$

whence, generally,

$$\text{R}\psi x = 1. \qquad (75)$$

For example, in the Calculus of H,

$$\text{G}x^{-1} = 1, \qquad (76)$$

as by (66). We may, indeed, derive (75) directly from (63).

29. If there is more than one independent variable, it is proper to write S_x, S_y, &c., the subscript letter denoting the variable with respect to which the operation is performed. Of two simple operations, S_x^m, S_y^n, performed successively on $\phi\,(x, y)$, it is a matter of indifference which comes first, the result in either case being $\phi\,(\psi^{-1}[\psi x + m],\ \psi^{-1}[\psi y + n])$; and it might readily be proved that all functions of two such independent operations are commutative.

30. If the operation S_yS_z be performed on a function of u and v, where u is a function of y and v a function of z, and if we then make y and z both equal to x, the result is the same as if we first make y and z equal to x, and then operate with S_x. The same remark applies to all powers, and, therefore, to all functions, of S_yS_z. If, instead of y, we write $x\,|\,u$, which may be interpreted "x varying only in u", and instead of z, $x\,|\,v$, "x varying only in v", the double operation $\text{S}_{x\,|\,u}\text{S}_{x\,|\,v}$ is the same as S_yS_z, and is equivalent to S_x. The symbol $\text{S}_{x\,|\,u}$ represents what may be called a partial operation, performed with

respect to x in u. If $u=x$, we shall have the symbol $s_{x\,|\,x}$, which is needlessly cumbrous in appearance, and may advantageously be replaced by the abbreviated form $s_{|\,x}$. In general, χ being any function,

$$\phi s_x \chi\,(u, v, w\,\ldots) = \phi\,(s_{x\,|\,u} s_{x\,|\,v} s_{x\,|\,w}\ldots)\,\chi\,(u, v, w\,\ldots)\,. \tag{77}$$

Also,

$$\phi s_{x\,|\,u} \chi\,(u, v) = \phi\,(s_x s_{x\,|\,v}{}^{-1})\,\chi\,(u, v)\,. \tag{78}$$

Substituting ϕ log for ϕ,

$$\phi R_x \chi\,(u, v, w\,\ldots) = \phi\,(R_{x\,|\,u} + R_{x\,|\,v} + R_{x\,|\,w} + \ldots)\,\chi\,(u, v, w\,\ldots)\,, \tag{79}$$

$$\phi R_{x\,|\,u} \chi\,(u, v) = \phi\,(R_x - R_{x\,|\,v})\,\chi\,(u, v)\,. \tag{80}$$

As special cases, among others,

$$R_x^n \chi\,(u, v, w\,\ldots) = R_{x\,|\,u} + R_{x\,|\,v} + R_{x\,|\,w} + \ldots)^n\,\chi\,(u, v, w\,\ldots)\,, \tag{81}$$

$$R_x uv = vR_x u + uR_x v\,. \tag{82}$$

31. If u is a function of x, any other function of x is of course a function of u, and may be operated upon by any function of s_u; but functions of s_u and functions of s_x are not usually commutative. It may be shown that s_u is equivalent to s'_x, where s' depends on ψ', and where u, ψ, and ψ' are so related that, when two are given, the third is determined by the equation

$$\psi u = \psi' x\,. \tag{83}$$

Starting with this equation, we have, successively,

$$\chi\psi\psi^{-1}\,(\psi u + n) = \chi\psi'\psi'^{-1}\,(\psi' x + n)\,, \tag{84}$$

$$s_u^n \chi\psi u = s_x'^n \chi\psi' x\,; \tag{85}$$

whence, since $\psi u = \psi' x$,

$$s_u^n = s_x'^n\,, \tag{86}$$

$$\phi s_u = \phi s'_x\,. \tag{87}$$

Thus, from (73), writing v for ϕx,

$$R_x v = R_x \psi u\,.\,R_u v\,. \tag{88}$$

32. The simplest Calculus is, of course, that in which $\psi x = x$, and $s\phi x = \phi\,(x+1)$. Here the operation s is that which I have called Enlargement, and is denoted by the symbol E. This calculus may, therefore, properly be called the Calculus of Enlargement. The most important function of E is $\log E = D$, which corresponds to R in the foregoing general discussion, whenever s is replaced by E.

33. In (73), let $\psi x = x$, and let us write ψ and R for ψ' and R'; then putting $v = \phi x$,

$$Dv = D\psi x\,.\,Rv\,, \tag{89}$$

$$\mathrm{R}v = \frac{\mathrm{D}v}{\mathrm{D}\psi x}, \tag{90}$$

$$\mathrm{R}u = \frac{\mathrm{D}u}{\mathrm{D}\psi x} = \frac{\mathrm{D}u}{\mathrm{D}v}\,\mathrm{R}v, \tag{91}$$

where, if $u = x$,

$$\mathrm{R}x = \frac{\mathrm{D}x}{\mathrm{D}\psi x}. \tag{92}$$

Again, from (73),

$$\mathrm{R}\phi x = \mathrm{R}x \,.\, \mathrm{D}\phi x\,. \tag{93}$$

34. It follows from (87) that all the processes of any calculus, say that of S', may be expressed in the language of any one given calculus, say that of S, by means of suitable artifices. It is therefore unnecessary to discuss in detail more than one of these systems; and the preference must naturally be given to the simplest of all, the Calculus of Enlargement. As a mere matter of interest, however, I shall, before closing this essay, make some suggestions concerning another calculus, comprising those operations which are functions of M, where $\psi x = \log x$, and

$$\mathrm{M}^h\phi x = \phi\,(x\varepsilon^h)\,. \tag{94}$$

This system may, in want of a better term, be called the Calculus of Multiplication.

III. *Theory of the Functions of* E.

35. The symbol E has sometimes been defined as ε^{D}, sometimes as $1 + \Delta$, and sometimes as representing an operation such that $\mathrm{E}\phi x = \phi\,(x + 1)$. It has also sometimes been used to denote the operation which changes ϕx into $\phi\,(x + h)$. We cannot now accept a definition in terms of Δ or D, for a simple operation ought not to be defined in terms of one more complex, nor can we agree that E shall be dependent on any arbitrary quantity h; $\mathrm{E}\phi x$ must be $\phi\,(x + 1)$ and nothing else. Yet if $\mathrm{E}\phi x = \phi\,(x + 1)$ express the definition of E, it will require considerable labor to prove that in all cases $\mathrm{E}^h\phi x = \phi\,(x + h)$, and then only when h expresses some positive or negative quantity; and the argument will not be free from ambiguity, since, for example, it might be hard to prove that $\mathrm{E}^{\frac{1}{2}}\phi x$ cannot be its own opposite, namely, $-\phi\left(x + \frac{1}{2}\right)$. I find it better to define E^h, like S^h, as a compound symbol representing that simple operation which changes ϕx into $\phi\,(x + h)$, whatever be the meaning of h. In this light we must regard E, when without an index, as an abbreviated

form of E^1. Since $cE^0\phi x = c\phi x$, we observe that $cE^0 = c$ and $E^0 = 1$, and hence that any constant may be regarded as a function of E of the form cE^0.

36. It would have been sufficient to define E^h as that special case of S^h where $\psi x = x$, and it may be said at once that all which has been shown to be true of S and its functions is true of E and its functions. If any one shall hereafter deem it best, for teaching the rudiments of the Calculus of Enlargement, to omit all mention of other possible systems, based on simple operations other than E, he will find it sufficient to say concerning E what has been said above concerning S, substituting x for ψx. It is not now necessary to repeat concerning E what has been proved in regard to all repetitive simple operations, and I shall confine my attention to certain properties pertaining to all functions of E as such. While nearly all of these properties are now no doubt first exhibited in this light, it will be seen that some of them are already known, more or less explicitly, as properties pertaining to algebraic functions of D. Such propositions will, however, be found to have been generalized, the properties hitherto known concerning algebraic functions of D being now exhibited concerning all functions of E, and therefore concerning all functions of D. It will be remarked that the theorems about to be stated regarding functions of E are developed more easily than if they were to be proved as relating to functions of D; particularly when the comparative ease with which E and D may be defined is taken into consideration, such definition being an essential element in either case.

37. If the general term of ϕx is $a_n x^n$, that of $\phi E_x \psi\,(x + y)$ is $a_n E_x^n \psi\,(x + y) = a_n \psi\,(x + n + y)$, supposing x and y to be independent, and this for the same reason is also the general term of $\phi E_y \psi\,(x + y)$, so that all terms correspond, and

$$\phi E_x \psi\,(x + y) = \phi E_y \psi\,(x + y)\,. \tag{95}$$

The same may be shown for any number of variables. Also,

$$\phi E_x \psi\,(x - y) = \phi\,(E_y^{-1})\,\psi\,(x - y)\,. \tag{96}$$

38. If the general term of ϕx is $a_n x^n$, and that of ψx is $b_m x^m$, the general term of $\phi E_x c^{xy}\,\psi\,(c^x)$ is $a_n E_x^n c^{xy} b_m c^{xm} = a_n b_m E_x^n c^{x(y+m)} = a_n b_m c^{(x+n)(y+m)}$. Similarly, $\psi E_y c_{xy}\,\phi\,(c^y)$ and $c^{xy}\,\phi\,(c^y E_x)\,\psi\,(c^x)$ will be found to have this same general term, so that

$$\phi E_x c^{xy}\,\psi\,(c^x) = \psi E_y c^{xy}\,\phi\,(c^y), \tag{97}$$

$$\phi E_x c^{xy}\,\psi\,(c^x) = c^{xy}\,\phi\,(c^y E_x)\,\psi)\,c^x), \tag{98}$$

$$\psi E_y c^{xy}\,\phi(c^y) = c^{xy}\,\phi\,(c^y E_x)\,\psi\,(c^x). \tag{99}$$

In general, for any number of independent variables,

$$\phi E_x \psi E_y \ldots \chi E_u \zeta E_v c^{xy \ldots uvw} \xi (c^{xy \ldots uv})$$
$$= \phi E_x \psi E_y \ldots \chi E_u \xi E_w c^{xy \ldots uvw} \zeta (c^{xy \ldots uw})$$
$$= c^{xy \ldots uvw} \psi (c^{x \ldots uvw} E_y) \ldots \zeta (c^{xy \ldots uw} E_v) \xi (c^{xy \ldots uv} E_w) \phi (c^{y \ldots uvw}), \quad (100)$$

where ϕ, ψ, &c., are arbitrary functions. Again, similarly,

$$\phi (c^y E_x^h) \psi (c^x) = \psi (c_x E_y^h) \phi (c^y). \quad (101)$$

Again, since $E_x^n c = E_x^n x^0 c = c$,

$$\phi E_x c = \phi 1 c. \quad (102)$$

Here c represents anything independent of x.

39. There are many special cases of the foregoing propositions which are themselves important general theorems. Some of these will now be mentioned. If in (95) and (96) we make $y = 0$, we shall have

$$\phi E \psi x = \phi E_0 \psi (x + 0), \quad (103)$$
$$\phi E \psi x = \phi (E_0^{-1}) \psi (x - 0); \quad (104)$$

and from the former of these, observing (102),

$$\phi E x = x \phi 1 + \phi E_0 0. \quad (105)$$

Again,

$$\phi E \sin x = \phi E_0 \sin (x + 0) = \sin x \phi E_0 \cos 0 + \cos x \phi E_0 \sin 0, \quad (106)$$
$$\phi E \cos x = \cos x \phi E_0 \cos 0 - \sin x \phi E_0 \sin 0. \quad (107)$$

It may be observed that since $\cos n = \cos (-n)$, $E_0^n \cos 0 = E_0^{-n} \cos 0$, and in general,

$$\phi E_0 \cos 0 = \phi (E_0^{-1}) \cos 0. \quad (108)$$

If, in (97), we make $y = 1$,

$$\phi E c^x \psi (c^x) = \psi (E_1) c^{x1} \phi (c^1), \quad (109)$$

and if $\psi (c^x) = 1$,

$$\phi E c^x = c^x \phi c. \quad (110)$$

If $y = 0$,

$$\phi E \psi (c^x) = \psi E_0 c^{x0} \phi (c^0), \quad (111)$$

where, if $\phi E = 1$,

$$\psi (c^x) = \psi E_0 c^{x0}, \quad (112)$$

which may be regarded as one form of Herschel's theorem. If, in (98), we write ψx for $\psi (c^x)$, and put $y = 1$, we shall have

$$\phi E c^x \psi x = c^x \phi (c E) \psi x. \quad (113)$$

If, in (99), $x = 1$, we shall have, writing x for y,

$$\psi E c^x \phi (c^x) = c^x \phi (c^x E_1) \psi (c^1). \quad (114)$$

Similarly, supposing $x = 0$ in (99), and writing x for y,

$$\psi E \phi (c^x) = \phi (c^x E_0) \psi (c^0). \quad (115)$$

For example, let $\psi\text{E} = \text{E}^h$; then

$$\phi\,(c^{x+h}) = \phi\,(c^x\text{E}_0)\,c^{h0}. \tag{116}$$

If, in (101), we put $y = 1$,

$$\phi\,(c\text{E}^h)\,\psi\,(c^x) = \psi\,(c^x\text{E}_1^h)\,\phi\,(c^1)\,, \tag{117}$$

while if $y = 0$,

$$\phi\,(\text{E}^h)\,\psi\,(c^x) = \psi\,(c^x\text{E}_0^h)\,\phi\,(c^0)\,. \tag{118}$$

In all these theorems, as well as in the more general ones from which they are derived, c may have any value. If we assign to it the value ε, we shall produce another series of theorems, for the most part less general in character, which it is not now necessary to write out in full.

40. If we have to do with two or more independent variables, we are at liberty to regard them as being themselves functions of a single supposed variable, which let us call l, the form of the functions being such that $x = gl + g'$, $y = kl + k'$, &c., where g, g', &c., are arbitrary constants; for independent variables may be viewed either as equicrescent quantities, in which case they must be functions, of the form mentioned, of some standard variable, or as quantities to which arbitrary values may be assigned, in which case, again, there is no difficulty in accepting the foregoing statement.* Since

$$\text{E}_l\phi x = \phi\,(gl + g + g') = \phi\,(x + g) = \text{E}_x^g\phi x\,, \tag{119}$$

E_l is a function of E_x, and all functions of E_l will be commutative with all functions of E_x. For E_l I shall hereafter use the symbol e, and for $\text{E}_{l\,|\,x}$, $\text{E}_{l\,|\,y}$, &c., the symbols e_x, e_y, &c.

IV. *Analytical Theory of Differentiation.*

41. Let $\text{D} = \log \text{E}$, and $d = \log e$. The former statement is new only as a definition, while the latter is, I suppose, novel in all respects.† Both D and d are functions of E, and have, therefore, all the properties which pertain to such functions in general. The operation denoted by D is Differentiation. That denoted by $d = \text{D}_l$ is in reality the same operation, performed with

* To quote language used by Lagrange on another subject, "quoique dans les fonctions de deux variables que nous considérons ici, les deux variables soient censées indépendantes, . . . rien n'empêche cependant qu'on ne puisse regarder ces variables elles-mêmes comme des fonctions d'une autre variable quelconque, mais fonctions indéterminées et arbitraires." *Calcul des Fonctions*, ed. 1806, p. 334.

† That is to say, taking e as it has just been defined, namely, as equivalent to E_l, the symbol of enlargement performed with respect to an assumed variable l, where l is such that $x = gl + g'$. Nevertheless, on the one hand, it is already not unusual to say that a differential may be regarded as a differential coefficient taken with respect to an assumed variable; and on the other hand, it has been noticed by Arbogast (*Calcul des Dérivations*, p. 376) that, using our notation, $d = \log \text{E}_x^g$, where g is any arbitrary constant. The present statement connects these two ideas, and indicates the form of the relation between l and x.

respect to an imagined variable. If it be desired in any case to make a verbal distinction between D and d, the operation denoted by d may be called "taking the differential"; but the word differentiation has been so long used in both senses, and the danger of misunderstanding is usually so slight, that such a verbal distinction will not often be required.

42. The resultant of the operation D is usually known by the term Differential Coefficient, though it is also sometimes called Derived Function or Derivative, and that of the operation d is known as a Differential. The term Derivative cannot be permanently satisfactory unless the word Derivation be substituted for Differentiation, a proposal which would not be listened to. It is on every account desirable that the operation and its resultant should have cognate names. The terms Derivative and Differential Coefficient are more or less objectionable, the one as recalling too strongly Lagrange's doctrine of Derived Functions, a theory not now in general use as an explanation of differentiation, the other as indicating a mere appendage to a differential; and the latter term is besides insufferably cumbrous. The word Differentiation, though introduced only in the present century into the language, is now firmly rooted. To express the resultant of this operation, and as a substitute for the phrase Differential Coefficient, I venture to coin the noun Differentiate. To this noun, as denoting that which has been differentiated, there seems to be no etymological objection, since it follows the analogy of such words as graduate, associate, duplicate, postulate, delegate, &c.

43. Just as a differential is in one sense a differentiate, since $d = \mathrm{D}_l$, so also in another sense may a differentiate be regarded as a differential, since, if we put $g = 1$, we have $\mathrm{D}_x = \log \mathrm{E}_x^g = \log e = d$. The differentiate is the simpler of the two, analytically, while the differential is frequently the more useful and intelligible for practical purposes. As both may be embraced in the same theory, there is no sufficient reason for excluding either from consideration. If the imagined variable l represents time, the differential is a differentiate with respect to time, and is known as a Fluxion. On the other hand, if, in $x = gl + g'$, we have g infinitesimal, the differential will also be infinitesimal, since $d = \log e = \log \mathrm{E}_x^g = g\mathrm{D}_x$.

44. From (69) we have Taylor's theorem,

$$\mathrm{E}^h = 1 + h\mathrm{D} + \frac{1}{2} h^2\mathrm{D}^2 + \frac{1}{2.3} h^3\mathrm{D}^3 + \ldots, \tag{120}$$

$$\phi\,(x + h) = \phi x + h\mathrm{D}\phi x + \frac{1}{2} h^2\mathrm{D}^2\phi x + \ldots. \tag{121}$$

Applied to $\phi 0$, (120) becomes Maclaurin's theorem; to x^m, the binomial theorem. The symbolic form (120) is always true, and the theorem itself (121) is therefore formally correct, though the resulting series is not always algebraically intelligible, and, even when intelligible, cannot, unless convergent, be verified arithmetically.* The following modification, possibly novel, will sometimes be found useful:

$$\phi(x+h)=\phi x+h\mathrm{D}\phi\left(x+\frac{1}{2}h\right)+\frac{h^3\mathrm{D}^3\varphi\left(x+\frac{1}{2}h\right)}{4.2.3}+\frac{h^5\mathrm{D}^5\varphi\left(x+\frac{1}{2}h\right)}{4^2.2.3.4.5}+\ldots \quad (122)$$

This is found by subtracting the development of $\phi\left(x-\frac{1}{2}h\right)$ from that of $\phi\left(x+\frac{1}{2}h\right)$ and then writing $x+\frac{1}{2}h$ for x; or, symbolically, from $\mathrm{E}^h=1+(\mathrm{E}^{\frac{1}{2}h}-\mathrm{E}^{-\frac{1}{2}h})\,\mathrm{E}^{\frac{1}{2}h}$. To extend Taylor's theorem to functions of two or more variables, we have only to develop $\mathrm{E}_x^h\mathrm{E}_y^k\ldots=\varepsilon^{h\mathrm{D}_x+k\mathrm{D}_y+\cdots}$.

45. It was shown in paragraph 12 that when $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots$, $x=y+\frac{1}{2}y^2+\frac{1}{2.3}y^3+\ldots$. A more direct proof of this reversion, which is a step in the demonstration of Taylor's theorem, is as follows. Log $[1+a+b\,(1+a)]$ is a certain series of powers of the expression $a+b\,(1+a)$. Expanding these powers, which are all positive and integral, by the binomial theorem, and separating the series forming the coefficient of $b^n\,(1+a)^n$, then expanding $(1+a)^n$ and multiplying the result by the coefficient just separated, and finally separating from the product the series forming the coefficient of a^mb^n, we find it to be, for all values of m and n greater than 0,

$$(-1)^{n-1}\frac{1}{m^{m)}}\left[(n-1)^{m-1)}-mn^{m-1)}\right.$$
$$\left.+\ldots(-1)^r\frac{m^{r)}}{r^{r)}}(n+r-1)^{m-1)}\ldots(-1)^m(n+m-1)^{m-1)}\right], \quad (123)$$

where $x^{k)}=x\,(x-1)\ldots(x-k+1)$. The series enclosed in brackets is, by a theorem in finite differences, equal to 0; hence all terms in a^mb^n, that is to say, all terms which contain both a and b, vanish. The terms remaining, which contain a alone and b alone, are respectively

$$a-\frac{1}{2}a^2+\frac{1}{3}a^3-\ldots=\log\,(1+a), \quad (124)$$

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*The expansions of $\varepsilon^{-\frac{1}{h}}$, $\varepsilon^{-\frac{1}{h^2}}$, &c., apparently convergent and untrue, are really divergent and unintelligible, as may be seen on examination of those of $\varepsilon^{-\frac{1}{x+h}}$, &c., when x is very small. The coefficient of h^n in each of the former expansions contains a term of the form $\infty^n\frac{\varepsilon^{-v}v^n}{2.3\ldots n}$, where v is infinite, a form which becomes $\sim^\infty$ when $n=v$.

$$b - \frac{1}{2}b^2 + \frac{1}{3}b^3 - \ldots = \log(1+b)\,. \tag{125}$$

Hence,

$$\log[1+a+b(1+a)] = \log[(1+a)(1+b)] = \log(1+a) + \log(1+b). \tag{126}$$

If $y = \log(1+x) = x - \frac{1}{2}x^2 + \ldots,$ let us assume, as the reverted series,

$$x = y + c_2y^2 + c_3y^3 + \ldots. \tag{127}$$

Similarly, if $v = \log(1+u)$,

$$u = v + c_2v^2 + c_3v^3 + \ldots. \tag{128}$$

Since, by (126), $y + v = \log[(1+x)(1+u)]$,

$$(1+x)(1+u) = 1 + (y+v) + c_2(y+v)^2 + c_3(y+v)^3 + \ldots. \tag{129}$$

But, from (127) and (128),

$$(1+x)(1+u) = (1+y+c_2y^2+\ldots)(1+v+c_2v^2+\ldots)\,. \tag{130}$$

Equating the coefficients of v, and then comparing those of y, y^2, etc., we find that $2c_2 = 1$, $3c_3 = c_2$, $4c_4 = c_3$, and so on; whence $c_2 = \frac{1}{2}$, $c^3 = \frac{1}{2.3}$, and so on. Here, throughout, x and y are symbols devoid of meaning.

46. From (75),

$$\mathrm{D}x = 1\,. \tag{131}$$

It follows that dx, which is equal to $g\mathrm{D}x$, is equal to g; that $y = k$, &c., showing that dx, dy, &c., are arbitrary constants when x, y, &c., are independent variables.

47. From (79),

$$\phi\mathrm{D}_x\psi(u, v, w, \ldots) = \phi(\mathrm{D}_{x\,|\,u} + \mathrm{D}_{x\,|\,v} + \mathrm{D}_{x\,|\,w} + \ldots)\,\psi(u, v, w, \ldots)\,, \tag{132}$$

a general theorem which, though I do not remember having seen it, may be already known. If $\phi\mathrm{D}_x = \mathrm{D}_x^n$, we derive the following theorem, substantially due to Arbogast,

$$\mathrm{D}_x^n\psi(u, v, w, \ldots) = (\mathrm{D}_{x\,|\,u} + \mathrm{D}_{x\,|\,v} + \ldots)^n\,\psi(u, v, w, \ldots)\,, \tag{133}$$

of which the next, known as Leibnitz's theorem, is a special case:

$$\mathrm{D}_x^n uv = (\mathrm{D}_{x\,|\,u} + \mathrm{D}_{x\,|\,v})^n uv\,. \tag{134}$$

If $n = 1$,

$$\mathrm{D}_x uv = \mathrm{D}_{x\,|\,u} uv + \mathrm{D}_{x\,|\,v} uv = v\mathrm{D}_x u + u\mathrm{D}_x v\,. \tag{135}$$

48. From (80), similarly,

$$\phi\mathrm{D}_{x\,|\,u}\psi(u, v) = \phi(\mathrm{D}_x - \mathrm{D}_{x\,|\,v})\,\psi(u, v)\,, \tag{136}$$

from which

$$\mathrm{D}_{x\,|\,u}{}^n\psi(u, v) = (\mathrm{D}_x - \mathrm{D}_{x\,|\,v})^n\psi(u, v)\,, \tag{137}$$

of which the following, ascribed by Price to Hargreave, is a special case:

$$\mathrm{D}_{x\,|\,u}{}^n uv = (\mathrm{D}_x - \mathrm{D}_{x\,|\,v})^n uv\,. \tag{138}$$

49. From (88),

$$\mathrm{D}_x v = \mathrm{D}_x u \,.\, \mathrm{D}_u v \,.^* \tag{139}$$

If $v = x$,

$$\mathrm{D}_x u \,.\, \mathrm{D}_u x = \mathrm{D} x = 1\,, \tag{140}$$

whence

$$\mathrm{D}_x u = \frac{1}{\mathrm{D}_u x}\,. \tag{141}$$

Again, from (139) and (141),

$$\mathrm{D}_x u = \frac{\mathrm{D}_x v}{\mathrm{D}_u v} = \frac{\mathrm{D}_v u}{\mathrm{D}_v x}\,. \tag{142}$$

If $v = l$, we have, since $d = \mathrm{D}_l$,

$$\mathrm{D}_x u = \frac{du}{dx}\,. \tag{143}$$

It follows that wherever D_x is written we may read $\frac{d}{dx}$, and *vice versa*; and from (141) we see that this is true whether x is or is not an independent variable. Again, from (139),

$$\mathrm{D}_{x\,|\,u} = \mathrm{D}_x u \,.\, \mathrm{D}_{|\,u}\,, \tag{144}$$

one of which expressions may always be replaced by the other.

50. All results obtained in the language of differentiates may of course be expressed at once, *mutatis mutandis*, in that of differentials, and *vice versa.* In the case of partial differentiation, the student should be informed that he will frequently meet with a certain ambiguous form of expression, which may be illustrated by saying that he will find $\mathrm{D}_{|\,x}^{\,2}\mathrm{D}_{|\,y} u$ written $\frac{d^3u}{dx^2dy}$. Perhaps it would be well to avoid this ambiguity in future by writing, for example, $\frac{d_x^2 d_y u}{dx^2dy}$, where d_x and d_y must be regarded as abbreviations of $\mathrm{D}_{l\,|\,x}$ and $\mathrm{D}_{l\,|\,y}$.

51. A large number of theorems relating to all functions of D may be derived at once from those already obtained concerning functions of E. Most of those which I now proceed to mention are known, though, as hitherto proved, they are known only for such forms of function as can be expressed in integral powers of D. In deriving these theorems from (95–102), it will be seen that changes are sometimes made in the form of expression, such as writing ϕ log for ϕ, ε^k for c, &c.

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*If I were writing an elementary treatise, I should introduce, at this point and elsewhere, the usual proofs and illustrations, together with others to be hereafter suggested. I have, for present convenience, deferred the consideration of such explanations of D as are afforded by vanishing fractions and series, but wish it to be understood that my separation of the "analytical" from the "explanatory theory of differentiation" is wholly arbitrary, and ought by no means to be imitated in any methodical treatise on the Calculus of Enlargement.

$$\phi \mathrm{D}_x \psi\,(x+y) = \phi \mathrm{D}_y \psi\,(x+y)\,, \tag{145}$$

$$\phi \mathrm{D}_x \psi\,(x-y) = \phi\,(-\mathrm{D}_y)\,\psi\,(x-y), \tag{146}$$

$$\phi \mathrm{D}_x \varepsilon^{kxy} \psi\,(kx) = \psi \mathrm{D}_y \varepsilon^{kxy} \phi\,(ky), \tag{147}$$

$$\phi \mathrm{D}_x \varepsilon^{mx} \psi x = \varepsilon^{mx} \phi\,(m + \mathrm{D}_x)\,\psi x, \tag{148}$$

$$\phi \mathrm{D}_y \varepsilon^{kxy} \psi\,(ky) = \varepsilon^{kxy} \psi\,(ky + \mathrm{D}_x)\,\phi\,(kx), \tag{149}$$

$$\begin{aligned}
&\phi \mathrm{D}_x \psi \mathrm{D}_y \ldots \chi \mathrm{D}_u \zeta \mathrm{D}_v \varepsilon^{kxy \ldots uvw} \xi\,(kxy \ldots uv) \\
&\quad = \phi \mathrm{D}_x \psi \mathrm{D}_y \ldots \chi \mathrm{D}_u \xi \mathrm{D}_w \varepsilon^{kxy \ldots uvw} \zeta\,(kxy \ldots uw) \\
&\quad = \varepsilon^{kxy \ldots uvw} \psi\,(kx \ldots uvw + \mathrm{D}_y) \ldots \xi\,(kxy \ldots uv + \mathrm{D}_w)\,\phi\,(ky \ldots uvw),
\end{aligned} \tag{150}$$

$$\phi\,(ky + h\mathrm{D}_x)\,\psi\,(kx) = \psi\,(kx + h\mathrm{D}_y)\,\phi\,(ky), \tag{151}$$

$$\phi \mathrm{D}_x c = \phi 0 c\,. \tag{152}$$

In all of these the variables are supposed to be independent. From (103–118) we have the following, which, although mere special cases of those just given, are still of great importance as general theorems:

$$\phi \mathrm{D} \psi x = \phi \mathrm{D}_0 \psi\,(x + 0)\,, \tag{153}$$

$$\phi \mathrm{D} \psi x = \phi\,(-\mathrm{D}_0)\,\psi\,(x - 0)\,, \tag{154}$$

$$\phi \mathrm{D} x = \phi 0 x + \phi \mathrm{D}_0 0\,, \tag{155}$$

$$\phi \mathrm{D} \sin x = \sin x \phi \mathrm{D} \cos 0 + \cos x \phi \mathrm{D} \sin 0\,, \tag{156}$$

$$\phi \mathrm{D} \cos x = \cos x \phi \mathrm{D} \cos 0 - \sin x \phi \mathrm{D} \sin 0\,, \tag{157}$$

$$\phi \mathrm{D} \cos 0 = \phi\,(-\mathrm{D}) \cos 0, \tag{158}$$

$$\phi \mathrm{D} \varepsilon^{kx} \psi\,(kx) = \psi \mathrm{D}_1 \varepsilon^{kx1} \phi\,(k1), \tag{159}$$

$$\phi \mathrm{D} \varepsilon^{kx} = \varepsilon^{kx} \phi k, \tag{160}$$

$$\phi \mathrm{D} \psi\,(kx) = \psi \mathrm{D}_0 \varepsilon^{kx0} \phi\,(k0)\,, \tag{161}$$

$$\phi x = \phi \mathrm{D}_0 \varepsilon^{x0}, \tag{162}$$

$$\phi \mathrm{D} \varepsilon^{kx} \psi\,(kx) = \varepsilon^{kx} \psi\,(kx + \mathrm{D}_1)\,\phi\,(k1)\,, \tag{163}$$

$$\phi \mathrm{D} \psi\,(kx) = \psi\,(kx + \mathrm{D}_0)\,\phi\,(k0)\,, \tag{164}$$

$$\phi\,(x + h) = \phi\,(x + \mathrm{D}_0)\,\varepsilon^{h0}, \tag{165}$$

$$\phi\,(k + h\mathrm{D})\,\psi\,(kx) = \psi\,(kx + h\mathrm{D}_1)\,\phi\,(k1)\,, \tag{166}$$

$$\phi\,(h\mathrm{D})\,\psi\,(kx) = \psi\,(kx + h\mathrm{D}_0)\,\phi\,(k0)\,. \tag{167}$$

From these, putting $k = 1$,

$$\phi \mathrm{D} \varepsilon^{x} \psi x = \psi \mathrm{D}_1 \varepsilon^{x1} \phi 1\,, \tag{168}$$

$$\phi \mathrm{D} \psi x = \psi \mathrm{D}_0 \varepsilon^{x0} \phi 0\,, \tag{169}$$

$$\phi \mathrm{D} \varepsilon^{x} \psi x = \varepsilon^{x} \psi\,(x + \mathrm{D}_1)\,\phi 1\,, \tag{170}$$

$$\phi \mathrm{D} \psi x = \psi\,(x + \mathrm{D}_0)\,\phi 0\,, \tag{171}$$

$$\phi\,(1 + h\mathrm{D})\,\psi x = \psi\,(x + h\mathrm{D}_1)\,\phi 1\,, \tag{172}$$

$$\phi\,(h\mathrm{D})\,\psi x = \psi\,(x + h\mathrm{D}_0)\,\phi 0\,. \tag{173}$$

The possible difficulty of expressing ϕD in terms of E cannot be urged as an objection to the foregoing deductions. We know that ϕD can be expressed in terms of D; that each such power of D can be expressed in terms of Δ, and again that each such power of Δ can be expressed in terms of E. Inasmuch as we know that ϕD can be expressed in terms of E, it is unnecessary to inquire the exact form of the expression.

52 From (152), where c is anything independent of x,

$$\text{D}_x c = 0\,. \tag{174}$$

Inversely, operating on both sides of this equation by D_x^{-1},

$$\text{D}_x^{-1} 0 = c\,. \tag{175}$$

Here is an operation which creates something out of nothing; and since we cannot tell what that something may be, the results of this operation, and of all other operations which have the same creative faculty, must be indeterminate. I presume that, in general, all functions of E which cannot be expressed in positive integral powers of Δ are productive of indeterminate results. If any such operation, say B, is performed on $\phi x = \phi x + 0$, it produces, in addition to what we may call the principal form of Bϕx, a complementary function of x, the coefficients of which may be assigned at will. In the case before us, we perceive that when the operation D^{-1}, called Integration, and usually represented by the sign $\int$, is performed, we must introduce a complementary constant before we can venture to interpret the result. It is unnecessary to say much in this essay regarding integration. We shall have occasion to use the well-known definite integral $\int_0^\infty \varepsilon^{-x} x^m dx = \Gamma\,(1 + m)$, of which the fuller formal description is $\text{D}^{-1}\varepsilon^{-x}x^m{}_{[x=\infty]} - \text{D}^{-1}\varepsilon^{-x}x^m{}_{[x=0]}$. The sign $\int$, as commonly used, may be considered as equivalent to D_l^{-1}, since

$$\int \phi x dx = \text{D}^{-1}\phi x = dx\,.\,\text{D}_l^{-1}\phi x = \text{D}_l^{-1}\phi x dx\,. \tag{176}$$

53. From (160),

$$\text{D}\varepsilon^{kx} = \varepsilon^{kx} k\,, \tag{177}$$

$$\text{D}\varepsilon^{x} = \varepsilon^{x}. \tag{178}$$

If $x = \varepsilon^\theta$, $\text{D}_\theta x = \varepsilon^\theta = x$, the reciprocal of which is $\text{D}_x\theta$, or

$$\text{D} \log x = x^{-1}. \tag{179}$$

Hence,

$$\text{D}x^m = \text{D}_x\theta\,.\,\text{D}_\theta x^m = x^{-1}.\, m\varepsilon^{m\theta} = mx^{m-1}. \tag{180}$$

From (158),

$$\text{D} \cos 0 = - \text{D} \cos 0 = 0\,. \tag{181}$$

By Maclaurin's theorem,

$$\sin x = x\text{D} \sin 0 + x^2\text{Q}\,, \textit{ suppose}; \tag{182}$$

whence, assuming D sin 0 finite, which can be proved from (157) when x is infinitesimal,

$$\frac{\sin 0}{0} = \mathrm{D} \sin 0\,, \tag{183}$$

and since, by trigonometry, $\dfrac{\sin 0}{0} = 1$,

$$\mathrm{D} \sin 0 = 1. \tag{184}$$

Then, from (156) and (157),

$$\mathrm{D} \sin x = \cos x\,, \tag{185}$$

$$\mathrm{D} \cos x = -\sin x\,. \tag{186}$$

54. The following series, the result of integration by parts, a method deducible, as usual, from (135), are well known:*

$$\Gamma(1+p) = \int_0^x \varepsilon^{-x} x^p dx + \int_x^\infty \varepsilon^{-x} x^p dx$$

$$= \varepsilon^{-x} x^p \Big[1 + \frac{x}{p+1} + \frac{x^2}{(p+1)(p+2)} + \ldots + p x^{-1} + p\,(p-1)\,x^{-2} + \ldots\Big]; \tag{187}$$

$$\varepsilon^x = \Big[1 + \frac{x}{p+1} + \ldots + p x^{-1} + \ldots\Big] \frac{x^p}{\Gamma(1+p)}\,. \tag{188}$$

Let us write $h\mathrm{D}$ for x, and its antilogarithm E^h for ε^x, and let us suppose the subject of operation to be ϕx. We shall then obtain the following *extension of Taylor's theorem:*

$$\begin{aligned}\phi\,(x+h) = \Big[1 + \frac{h\mathrm{D}}{p+1} + \frac{h^2\mathrm{D}^2}{(p+1)(p+2)} + \ldots + p h^{-1}\mathrm{D}^{-1} \\ + p\,(p-1)\,h^{-2}\mathrm{D}^{-2} + \ldots\Big] \frac{h^p\mathrm{D}^p}{\Gamma(1+p)}\,\phi x\,.\end{aligned} \tag{189}$$

The original series terminates, in one direction, when p is an integer, so that in that case our extended theorem takes the usual form of Taylor's theorem. It will be observed that p may be any quantity except a negative integer. If $\phi x = x^m$, we shall have, as a special case, the extended binomial theorem of Roberts.† If we write 0 for x and x for h, we shall derive the following *extension of Maclaurin's theorem:*

$$\phi x = \Big[1 + \frac{x\mathrm{D}_0}{p+1} + \ldots + p x^{-1}\mathrm{D}_0^{-1} + \ldots\Big] \frac{x^p\mathrm{D}_0^p}{\Gamma(1+p)}\,\phi 0\,. \tag{190}$$

* De Morgan, *Calculus*, p. 590; Roberts, *Quarterly Journal*, VII, p. 207.

† In this, as in (189), when p is an integer, there can be no powers of h with negative indices. Unaware of this limitation, Roberts obtained anomalous and perplexing results. How near he came to formulating the extension of Taylor's theorem may be seen from the following quotations: "Let $\phi\,(x)$ be any function of x, then $\dfrac{\mathrm{D}_x^n}{\Gamma(1+n)}\,\phi\,(x) =$ coefficient of a development of $\phi\,(x+h)\ldots[k']$, according to powers of h to the base under n." "All that the equivalence $[k']$ means is this: if $\phi\,(x+h)$ can be developed according to powers of $(x+h)$, $\dfrac{\mathrm{D}_x^n}{\Gamma(1+n)}\,\phi\,(x)\,[\phi\,(x)$ being similarly developed] will give the corresponding coefficient of h^n in a development to the base index n."

While there may not now appear any practical use to which these theorems for expansion in fractional powers can be put, they will at least be found to throw some light on the theory of the subject. Various interesting series, such as for sin x, cos x, and of course ε^x, may be obtained by the use of (190), and x^m can be treated by it, when m is fractional, whereas Maclaurin's theorem cannot be employed in that case; the result of such treatment being that all terms except x^m vanish.

55. By employing (188) to expand ε^{x0} in (162), we have at once this *extension of Herschel's theorem*,

$$\phi x = \phi \mathrm{D}_0 \left[1 + \frac{x0}{p+1} + \ldots + px^{-1}0^{-1} + \ldots\right] \frac{x^p 0^p}{\Gamma(1+p)}. \tag{191}$$

Here, as before, p cannot be a negative integer.

56. I shall give more space to the consideration of the form of $\mathrm{D}^n x^m$, where n is fractional, than would be necessary were it not for the fact that it has been the subject of a noted controversy. Messrs. Liouville, Kelland and others make

$$\mathrm{D}^n x^m = (-1)^n \frac{\Gamma(-m+n)}{\Gamma(-m)} x^{m-n}, \tag{192}$$

while Peacock makes

$$\mathrm{D}^n x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-n)} x^{m-n}. \tag{193}$$

De Morgan (*Calculus*, p. 599) conjectures that "neither system has any claim to be considered as giving *the* form of $\mathrm{D}^n x^m$, though either may be *a* form." Later, Roberts shows, by strong arguments of analogy, that Peacock's form is tenable, while he admits the force of the arguments adduced in favor of that of Liouville. The reader cannot probably find in existence a more complete illustration of the difficulty with which such a subject is handled, under the indirect theory of differentiation heretofore followed, than that furnished by Roberts' argument. It is not too much to say that under that theory the meaning of D^n, where n is fractional, can only be guessed at. That indirect theory gives us D, the special case, and permits us to divine, if we can, by induction, analogy, or conjecture, the meaning of D^n, the general form. This is in every science the natural order of things so long as the general law, which shall furnish direct deductive proof, is unknown. The method now presented enables us to treat this case, like all others, with confidence and certainty. It makes us acquainted with D^n as one of many functions of E, and enables us to discuss, if we please, the general form D^n before the special form

D. If, in pursuance of the direct method, we arrive in any case at results which are not intelligible, we can only seek further for such expressions as we can understand, knowing that when found we can depend upon their accuracy, provided due allowance be made for possible complementary functions. I shall now try not only to show that Peacock's form is the principal form of $\text{D}^n x^m$, but also to indicate the precise nature of the error made by his antagonists.

57. If, in (189), $\phi x = x^p$, we derive a development of $(x+h)^p$ in powers of h which, when p is a fraction, extends to infinity in both directions. If $\text{D}^p x^p$ is a constant, the coefficients of h^{p+1}, h^{p+2}, &c., which are derived from $\text{D}^p x^p$ by differentiation, will vanish. That $\text{D}^p x^p$ is a constant may be shown from (173), which gives, writing z for h,

$$\text{D}^p x^p = z^{-p}\,(x + z\text{D}_0)^p 0^p, \tag{194}$$

wherein putting $z = x$ eliminates x. Omitting the vanishing terms of the development of $(x+h)^p$, and comparing the coefficient of h^p in the remainder of the development, namely, $\dfrac{\text{D}^p x^p}{\Gamma(1+p)}$, with that of h^p in the known development of $(x+h)^p$ by the binomial theorem, namely, 1, we have

$$\text{D}^p x^p = \Gamma\,(1+p)\,. \tag{195}$$

This equation is thus shown to be true for all cases except when p is a negative integer. That it is formally true in that case also may be seen upon repeated integration, resulting in a term containing $\text{D}^{-1}x^{-1} = \log x = \dfrac{x^0}{0} - \dfrac{.1}{0}$, the latter fraction being the complementary constant. Therefore, in all cases

$$\frac{\text{D}^m x^m}{\Gamma(1+m)} = \frac{\text{D}^{m-n} x^{m-n}}{\Gamma(1+m-n)}. \tag{196}$$

Operating on both sides with D^{n-m}, and multiplying by $\Gamma\,(1+m)$, we have, for all values of n, Peacock's formula,

$$\text{D}^n x^m = \frac{\Gamma(1+m)}{\Gamma(1+m-n)}\,x^{m-n}. \tag{197}$$

It may readily be shown that in this case there are no complementary terms in x^{m-n-1}, x^{m-n-2}, &c., such as might be created from 0 by the operation D^{n-m}. For, by (189), the coefficient of h^n in $(x+h)^m$ is $\dfrac{\text{D}^n x^m}{\Gamma(1+n)}$, while we know, by the common expansion of $(x+h)^n(x+h)^{m-n} = (h^n + \ldots)(x^{m-n} + \ldots)$, that this coefficient contains no other power of x than x^{m-n}.

58. The arguments by which it has been proved that Liouville's form of $\mathrm{D}^n x^m$ is correct have never been impugned, nor do I now impugn them, though I hold that it is not the principal form. If the performance of an ambiguous operation (see paragraph 52) such as D^n, where n is fractional, produces in one way a real result and in another an imaginary result differing from the real by a complementary function which D^n produces from 0, and which the inverse operation D^{-n} will reduce to 0, we are bound to accept the real result as the principal form. The proof of Liouville's formula depends on this equation, derived from (160),

$$\mathrm{D}^n \varepsilon^{-xv} = (-v)^n \varepsilon^{-xv}. \tag{198}$$

When n is fractional, this expression is imaginary. It is, however, formally correct, and no one seems to have suspected that another and real expression can be found. Even Roberts explicitly lays down $\mathrm{D}^n \varepsilon^{ax} = a^n \varepsilon^{ax}$, without limitation, as if it were the principal, or indeed the only possible, form. I shall now show that (198) is not the principal form of $\mathrm{D}^n \varepsilon^{-xv}$. Since $\varepsilon^{-xv} = x^0 v^0 - xv + \frac{x^2 v^2}{2} - \ldots$, we have, by (197),

$$\mathrm{D}^n \varepsilon^{-xv} = \frac{x^{-n}}{\Gamma(1-n)} - \frac{x^{1-n} v}{\Gamma(2-n)} + \frac{x^{2-n} v^2}{\Gamma(3-n)} - \ldots, \tag{199}$$

a series not only not imaginary, but also essentially convergent, and, therefore, eminently acceptable. If, on the other hand, we expand $(-v^n)\varepsilon^{-xv}$ by (188), writing $-xv$ for x, and $-n$ for p, we shall have the same series, with these additional terms, $-\frac{x^{-n-1} v^{-1}}{\Gamma(-n)} + \frac{x^{-n-2} v^{-2}}{\Gamma(-n-1)} - \ldots$. Now the additional terms become ultimately all of the same sign, forming a series infinite in value, as might have been expected from the imaginary character of the function developed; but it is especially to be remarked that they all vanish when operated upon by D^{-n}, showing that they constitute a complementary function, and are not necessarily part of the principal form of $\mathrm{D}^n \varepsilon^{-xv}$. Here then are two forms of $\mathrm{D}^n \varepsilon^{-xv}$, (198) and (199), one imaginary, the other real, the former being composed of the real form plus a complementary function. The real form is therefore the principal one. It is, however, only by employing the imaginary form that the expression given for $\mathrm{D}^n x^m$ by Liouville can be proved.

59. A good illustration of the ease with which secondary forms of such expressions as $\mathrm{D}^n x^m$ may be obtained consists in the application of (171) to $\mathrm{D}^n x^m$, whence

$$\mathrm{D}^n x^m = (x + \mathrm{D}_0)^m 0^n. \tag{200}$$

Now this binomial may be so expanded by Roberts' theorem as to produce a result differing from (197) by only a complementary function; but if on the other hand it is expanded in the usual way, in positive integral powers of D_0, it produces an expression, probably new,

$$\text{D}^n x^m = (x^m + mx^{m-1}\text{D}_0 + \ldots)\, 0^n, \tag{201}$$

which, although formally correct, can have no claim to be considered the principal form of $\text{D}^n x^m$. It does, however, give correct real results when n is a positive integer, and in every case satisfies, as does Liouville's expression, the requirement of interpolation of form. For example,

$$\text{D}^{\frac{1}{3}} x = x0^{\frac{1}{3}} + \frac{1}{3}\, 0^{-\frac{2}{3}} x^0, \tag{202}$$

and repeating,

$$\text{D}^{\frac{1}{3}}\text{D}^{\frac{1}{3}} x = x0^{\frac{2}{3}} + \frac{2}{3}\, 0^{-\frac{1}{3}} x^0, \tag{203}$$

$$\text{D}^{\frac{1}{3}}\text{D}^{\frac{1}{3}}\text{D}^{\frac{1}{3}} x = \text{D}x = x0 + \frac{3}{3}\, 0^0 x^0 = 1\,. \tag{204}$$

V. *Explanatory Theory of Differentiation.*

60. Although the various theorems of the Differential and Integral Calculus may readily be derived from the propositions already laid down, we have really taken but a narrow view of the subject. We have not done much more than to exhibit Taylor's theorem, and to ascribe to D as a function of E certain properties pertaining to such functions in general. We have now to examine more closely into the nature of the operation of differentiation, as disclosed by its symbolic definition, $\text{D} = \log \text{E}$.*

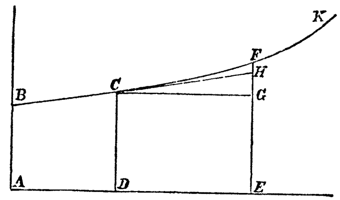
61. The coefficient of h in the expansion of $\phi\,(x + h)$ by Taylor's theorem is $\text{D}\phi x$. If, therefore, we know the development of any function of $x + h$ in positive integral powers of h, we know at once the differentiate of the same function of x. Thus, from the binomial theorem, we have $\text{D}x^m = mx^{m-1}$; from the exponential theorem, $\text{D}\varepsilon^x = \varepsilon^x$; from the logarithmic series, $\text{D} \log x = x^{-1}$; and from the trigonometrical series, $\text{D} \sin x = \cos x$ and $\text{D} \cos x = -\sin x$. This method of determining $\text{D}\phi x$ rests on surer grounds than the somewhat

*I must again observe that the order in which, for present convenience, these several matters are discussed is not that which should be followed in a methodical treatise on the Calculus of Enlargement. In such a work, the elementary explanations which we have now to consider should be introduced as soon as practicable after the first mention of differentiation, and be followed up at every convenient point by suitable illustrations.

similar principle underlying the *Calcul des Fonctions* of Lagrange, for we have what he had not, a symbolic demonstration of Taylor's theorem; and, not to dwell too long upon it, we may pass it by with the remark that in all probability no insuperable objection can be made to it.

62. Perhaps it has not been noticed hitherto that a simple variation of Taylor's theorem,

$$\Delta = \mathrm{D} + \frac{1}{2}\,\mathrm{D}^2 + \frac{1}{2\,.\,3}\,\mathrm{D}^3 + \ldots, \tag{205}$$



is remarkably susceptible of geometric illustration. For example, let $\mathrm{AD} = x$, and let the space ABCD, included between the straight line AD, the curve BCK, and the two perpendiculars AB and DC, be called ϕx. Take $\mathrm{DE} = 1$, and draw the lines EF perpendicular, and CG parallel, respectively, to AE; also CH tangent to the curve. Then,

$$\mathrm{D}\phi x = \mathrm{DEGC}, \tag{206}$$

$$\frac{1}{2}\,\mathrm{D}^2\phi x = \mathrm{CGH}, \tag{207}$$

$$\frac{1}{2\,.\,3}\,\mathrm{D}^3\phi x + \ldots = \mathrm{CHF}, \tag{208}$$

$$\Delta\phi x = \mathrm{CDEF} = \mathrm{D}\phi x + \frac{1}{2}\,\mathrm{D}^2 x + \frac{1}{2\,.\,3}\,\mathrm{D}^3 x + \ldots. \tag{209}$$

63. It is desirable to find as many expressions as possible for $\log x$ in terms either of x or of simple functions of x, and in them to write E for x, in order to arrive at the clearest understanding of the operation $\mathrm{D} = \log \mathrm{E}$ by attentive observation of its various algebraic equivalents. For this purpose the two general series (30, 31) presented in the foregoing Theory of Logarithms afford ample means.

64. From (30),

$$\mathrm{D} = \frac{1}{n}\,\mathrm{E}^n - \frac{1}{2n}\,\mathrm{E}^{2n} + \ldots - \frac{1}{n}\,\mathrm{E}^{-n} + \frac{1}{2n}\,\mathrm{E}^{-2n} - \ldots, \tag{210}$$

where n may have any value. If $n = 1$,

$$\mathrm{D} = \mathrm{E} - \frac{1}{2}\,\mathrm{E}^2 + \ldots - \mathrm{E}^{-1} + \frac{1}{2}\,\mathrm{E}^{-2} - \ldots. \tag{211}$$

Applied to ϕx,

$$\mathrm{D}\phi x = \frac{1}{n}\,[\phi\,(x+n) - \phi\,(x-n)] - \frac{1}{2n}\,[\phi\,(x+2n) - \phi\,(x-2n)] + \ldots, \tag{212}$$

$$\text{D}\phi x = [\phi(x+1) - \phi(x-1)] - \frac{1}{2}[\phi(x+2) - \phi(x-2)] + \ldots. \tag{213}$$

These series, which are probably new, will, owing to their symmetry of form, be readily borne in mind. To illustrate their use, let $\phi x = a^x$, and we have

$$\text{D}a^x = a^x\left[\frac{1}{n}(a^n - a^{-n}) - \frac{1}{2n}(a^{2n} - a^{-2n}) + \ldots\right] = a^x \log a. \tag{214}$$

Let $\phi x = x^m$, then

$$\text{D}x^m = [(x+1)^m - (x-1)^m] - \frac{1}{2}[(x+2)^m - (x-2)^m] + \ldots. \tag{215}$$

Here all terms in x^m, x^{m-2}, x^{m-4}, &c., obviously vanish. The terms in x^{m-3}, x^{m-5}, &c., contain, in the coefficient of each such power, a factor of the form $1 - 2^r + 3^r - \ldots$, where r is an even positive integer, so that, by a known theorem in finite differences, these terms likewise vanish. There remain the terms in x^{m-1}, whose coefficients are $2m - 2m + 2m - \ldots = m$, so that, finally,

$$\text{D}x^m = mx^{m-1}. \tag{216}$$

Again, putting $n = \dfrac{\pi}{2}$,

$$\begin{aligned} \text{D}\sin x &= \frac{2}{\pi}\left[\sin\left(x + \frac{\pi}{2}\right) - \sin\left(x - \frac{\pi}{2}\right) - \ldots\right] \\ &= \frac{2}{\pi}\left(2 - \frac{2}{3} + \frac{2}{5} - \ldots\right)\cos x = \cos x. \end{aligned} \tag{217}$$

It is needless, however, to multiply illustrations which will readily occur to the reader.

65. Let the symbol ∂ represent the operation of obtaining the ratio of the most general form of difference of a function to the corresponding difference of its variable; that is to say, let

$$\partial = \frac{\text{E}^{-ha+h} - \text{E}^{-ha}}{h},* \tag{218}$$

$$\partial\phi x = \frac{\varphi(x - ha + h) - \varphi(x - ha)}{h}. \tag{219}$$

The constants a and h may have any value, so that there will be an unlimited number of special cases, some of which will, from their greater importance, require distinct symbols. Thus, when $h = 1$ and $a = 0$, $\partial = \text{E} - 1 = \Delta$; when $h = 1$ and $a = 1$, $\partial = 1 - \text{E}^{-1}$, which is sometimes denoted by $`\Delta$; when $h = 1$ and $a = \dfrac{1}{2}$, $\partial = \text{E}^{\frac{1}{2}} - \text{E}^{-\frac{1}{2}}$, which let us represent by Λ; and

* Results more symmetrical, though less simple, can be got by writing $\dfrac{1}{2}(1 - b)$ for a.

when $h=0$ and $a=0$, $\partial=\mathrm{D}$. It will shortly be shown that we need not restrict D to the case where $a=0$, but that $\partial=\mathrm{D}$ when $h=0$, whatever be the (finite) value of a.

66. We now derive at once from (31) the following general *differentiate-expression:*

$$\mathrm{D}=\partial+\frac{2a-1}{2}h\partial^2+\frac{3a-1}{2}\,\frac{3a-2}{3}h^2\partial^3+\frac{4a-1}{2}\,\frac{4a-2}{3}\,\frac{4a-3}{4}h^3\partial^4+\ldots. \quad (220)$$

Of all special cases of this theorem, those are particularly important in which $h=0$ or $h=1$. When $h=0$, there are four chief cases, where $a=0$, $a=1$, $a=\frac{1}{2}$, and $a=\infty$, respectively; and when $h=1$, there are three chief cases, where $a=0$, $a=1$, and $a=\frac{1}{2}$, respectively. I shall discuss these in order.

67. Let $h=0$. In this case the general theorem is reduced to a vanishing fraction. Concerning vanishing fractions in general, it may be said that they are rendered needlessly obscure by presentation in the form $\frac{0}{0}$. Whenever we have to write 0 as the denominator of a fraction we ought, I think, if convenient, to express the numerator as a function of the denominator, or, in other words, as a function of 0, that symbol, when employed in the numerator, representing the denominator and nothing else. So expressed, it is impossible for a vanishing fraction to be ambiguous in meaning, supposing it possible to expand the numerator in positive integral powers of 0. It matters little whether such fractions are philosophically explained by the doctrine of infinitesimals or by that of limits. All that is necessary to their acceptance is to persuade ourselves in some way that $\frac{x}{x}=1$ when $x=0$.

68. When $h=0$, we have, therefore,

$$\mathrm{D}=\frac{\mathrm{E}^{(1-a)0}-\mathrm{E}^{-a0}}{0}, \quad (221)$$

an expression which may be instantly verified by expansion. It may, indeed, be shown that

$$\mathrm{D}=\frac{\mathrm{P}^0(\mathrm{E}^0-1)}{0}, \quad (222)$$

where P is any function of E. Of this (221) is a special case. In practice, we may apply (221) thus,

$$\mathrm{D}\phi x=\frac{\varphi(x-ha+h)-\varphi(x-ha)}{h}_{[h=0]}. \quad (223)$$

The differentiate of any function, therefore, is equal to an infinitely small difference of the function divided by the corresponding difference of the variable; or, in other words, to the limit of the ratio of differences indefinitely reduced. In this statement it will be observed that the word Difference cannot be replaced by the word Increment without obscuring the truth which is conveyed. To illustrate (223), let $a = -4$; then

$$\mathrm{D}x^3 = \frac{(x+5h)^3 - (x+4h)^3}{h}_{[h=0]} = 3x^2, \tag{224}$$

$$\mathrm{D}\varepsilon^x = \frac{\varepsilon^{x+5h} - \varepsilon^{x+4h}}{h}_{[h=0]} = \varepsilon^x. \tag{225}$$

69. Since $\mathrm{D} = \dfrac{d}{dx}$, where dx is an arbitrary constant, let $dx = h$; then, when h is infinitesimal,

$$d = \mathrm{E}^{(1-a)0} - \mathrm{E}^{-a0}. \tag{226}$$

This is to be interpreted as an order to perform the operation $\mathrm{E}^{(1-a)h} - \mathrm{E}^{-ah}$, to make $h = 0$, and to represent the 0 in question by the symbol dx. Applied to ϕx, it becomes

$$d\phi x = \phi\,(x - adx + dx) - \phi\,(x - adx)\,. \tag{227}$$

When, again, instead of being infinitesimal, dx is taken to have some tangible finite value, dx and $d\phi x$ have nevertheless the same ratio as if both were infinitely small, so that when dx is assigned, and the ratio ascertained, the value of $d\phi x$ is known. The doctrine of fluxions is a case in point.

70. If, in (221), $a = 0$,

$$\mathrm{D} = \frac{\mathrm{E}^0 - 1}{0}. \tag{228}$$

This is the symbolic embodiment, possibly not new, of the usual expression

$$\mathrm{D}\phi x = \frac{\varphi\,(x+h) - \varphi x}{h}_{[h=0]}. \tag{229}$$

If $a = 1$,

$$\mathrm{D} = \frac{1 - \mathrm{E}^{-0}}{0}, \tag{230}$$

$$\mathrm{D}\phi x = \frac{\varphi x - \varphi\,(x-h)}{h}_{[h=0]}, \tag{231}$$

the latter again being a known form. If $a = \dfrac{1}{2}$,

$$\mathrm{D} = \frac{\mathrm{E}^{\frac{0}{2}} - \mathrm{E}^{-\frac{0}{2}}}{0}, \tag{232}$$

$$\mathrm{D}\phi x = \frac{\varphi\left(x + \frac{1}{2}h\right) - \varphi\left(x - \frac{1}{2}h\right)}{h}_{[h=0]}, \tag{233}$$

both of which expressions are probably new. The three forms thus derived by making $a = 0$, $a = 1$, and $a = \frac{1}{2}$, may be called the upper, lower, and central vanishing fractions respectively. Correspondingly, from (226) and (227),

$$d = \mathrm{E}^0 - 1, \tag{234}$$

$$d\phi x = \phi\,(x + dx) - \phi x; \tag{235}$$

$$d = 1 - \mathrm{E}^{-0}, \tag{236}$$

$$d\phi x = \phi x - \phi\,(x - dx); \tag{237}$$

$$d = \mathrm{E}^{\frac{0}{2}} - \mathrm{E}^{-\frac{0}{2}}, \tag{238}$$

$$d\phi x = \phi\left(x + \frac{1}{2}\,dx\right) - \phi\left(x - \frac{1}{2}\,dx\right). \tag{239}$$

Of these, (235) and (237) are known forms.

71. Of the three chief vanishing fractions, with the expressions corresponding to them just given, the upper fraction will no doubt in most cases be found the most useful in practice, as being, on the whole, the simplest. Nevertheless, the central fraction (233) and the corresponding differential expression (239) will be found well worthy of attention on account of their symmetrical form. It cannot be doubted that cases will arise in which this quality of symmetry will prove an important aid to the analyst. To illustrate another advantage possessed by the central formulæ, let it be required to find $d\,(x^3)$. By the usual method,

$$d\,(x^3) = 3x^2dx + 3x\,(dx)^2 + (dx)^3, \tag{240}$$

and by the central method,

$$d\,(x^3) = 3x^2dx + \frac{1}{4}\,(dx)^3. \tag{241}$$

Here there is obviously less to be disregarded, and so far there is an advantage, even though it be only in appearance. Apart from all practical advantages, however, the consideration of the central formulæ cannot but be useful in affording a broader view of the subject than that usually taken. The same remark applies, of course, with still greater force to the general formulæ of paragraphs 68 and 69, not to speak of others still to be presented.

72. In the special cases thus far examined of the general differentiate-expression (220), we have supposed $h = 0$, with a finite. Let us now consider the case in which $h = 0$ and a is infinite. Let $a = -\frac{c}{h}$, so that $\partial = \mathrm{E}^c\,\frac{\mathrm{E}^h - 1}{h}{}_{[h=0]} = \mathrm{DE}^c$, where c has any finite value other than 0. Then $ah\partial = -\,c\mathrm{DE}^c$, and we have the following series,

$$\text{D} = \text{DE}^c - c\text{D}^2\text{E}^{2c} + \frac{3}{2}\, c^2\text{D}^3\text{E}^{3c} - \frac{4^2}{2\,.\,3}\, c^3\text{D}^4\text{E}^{4c} + \frac{5^3}{2\,.\,3\,.\,4}\, c^4\text{D}^5\text{E}^{5c} - \ldots, \tag{242}$$

$$\text{D}\phi x = \text{D}\phi\,(x + c) - c\text{D}^2\phi\,(x + 2c) + \frac{3}{2}\, c^2\text{D}^3\phi\,(x + 3c) - \ldots. \tag{243}$$

As special cases,

$$\text{D}\phi x = \text{D}\phi\,(x + 1) - \text{D}^2\phi\,(x + 2) + \frac{3}{2}\,\text{D}^3\phi\,(x + 3) - \ldots, \tag{244}$$

$$\text{D}\phi x = \text{D}\phi\,(x - 1) + \text{D}^2\phi\,(x - 2) + \frac{3}{2}\,\text{D}^3\phi\,(x - 3) + \ldots. \tag{245}$$

If we divide both members of (242) by DE^c, and, putting $h = -c$, operate on ϕx, also on $\phi 0$, we shall have

$$\phi\,(x + h) = \phi x + h\text{D}\phi\,(x - h) + \frac{3}{2}\, h^2\text{D}^2\phi\,(x - 2h) + \ldots, \tag{246}$$

$$\phi h = \phi 0 + h\phi'\,(-h) + \ldots, \tag{247}$$

where $\phi' x = \text{D}\phi x$. Though interesting, and probably new, these various series are comparatively unimportant.

73. Much more worthy of attention are those series, expressing the differentiate in terms of finite differences, which are derived from the general differentiate-expression (220) by giving to h some value other than 0. The principal value which h may assume is 1, and the formulæ derived for that value can be made to yield, by a suitable alteration of the variable, all the results obtainable by assigning to h any other value. When $h = 1$, we have the following general theorem for expressing a differentiate in terms of differences:

$$\text{D} = \partial + \frac{2a - 1}{2}\,\partial^2 + \frac{3a - 1}{2}\,\frac{3a - 2}{3}\,\partial^3 + \ldots. \tag{248}$$

Here $\partial = \text{E}^{1-a} - \text{E}^{-a}$, and $\partial\phi x = \phi\,(x - a + 1) - \phi\,(x - a)$. For example,

$$\partial c^x = c^x\,(c^{1-a} - c^{-a}) = c^x z\,,\ \textit{suppose}, \tag{249}$$

$$\partial^2 c^x = c^x z^2, \tag{250}$$

$$\text{D}c^x = c^x\left(z + \frac{2a - 1}{2}\, z^2 + \frac{3a - 1}{2}\,\frac{3a - 2}{3}\, z^3 + \ldots\right) = c^x \log c\,, \tag{251}$$

by (43).

74. In this case again, as with the general vanishing fraction (221), we find three principal values for a, namely, $a = 0$, $a = 1$, and $a = \frac{1}{2}$. Substituting these values respectively, we obtain three series, all more or less well known, expressing a differentiate in terms of what we may call upper, lower, and central differences. These are,

$$\text{D} = \Delta - \frac{1}{2}\,\Delta^2 + \frac{1}{3}\,\Delta^3 - \ldots, \tag{252}$$

$$\mathrm{D} = {}^{\backprime}\Delta + \frac{1}{2}\,{}^{\backprime}\Delta^2 + \frac{1}{3}\,{}^{\backprime}\Delta^3 + \ldots, \tag{253}$$

$$\mathrm{D} = \Lambda - \frac{\Lambda^3}{8.3} + \frac{3}{2}\,\frac{\Lambda^5}{8^2.5} - \frac{3.5}{2.3}\,\frac{\Lambda^7}{8^3.7} + \frac{3.5.7}{2.3.4}\,\frac{\Lambda^9}{8^4.9} - \ldots. \tag{254}$$

There is also a known series expressing D in terms of mean central differences, which may be derived as follows from (254). Let $\mathrm{I} = \frac{1}{2}(\mathrm{E}^{\frac{1}{2}} + \mathrm{E}^{-\frac{1}{2}})$; then $\mathrm{I} = (1 + \frac{1}{4}\Lambda^2)^{\frac{1}{2}}$, and, by expansion,

$$\mathrm{D}\mathrm{I}^{-1} = \left(\Lambda - \frac{\Lambda^3}{8.3} + \ldots\right)\left(1 - \frac{\Lambda^2}{8} + \ldots\right) = \Lambda - \frac{\Lambda^3}{6} + \frac{\Lambda^5}{30} - \ldots, \tag{255}$$

and

$$\mathrm{D} = \mathrm{I}\Lambda - \frac{\mathrm{I}\Lambda^3}{6} + \frac{\mathrm{I}\Lambda^5}{30} - \frac{\mathrm{I}\Lambda^7}{140} + \ldots. \tag{256}$$

75. The principal use to which these series have hitherto been put is to determine the value of a differentiate from given values of the function differentiated. The simplest possible illustration is probably as follows. Let us first construct a table of the values of x^2, and of their differences, from $x = -1$ to $x = 3$.

x	x^2	∂x^2	$\partial^2 x^2$	$\partial^3 x^2$
3	9		2	
		5		0
2	4	[4]	2	[0]
		3		0
1	1		2	
		1		0
0	0		2	
		-1		0
-1	1		2	

We see that $\Delta(-1)^2 = -1$, $\Delta^2(-1)^2 = 2$, $\Delta^3(-1)^2 = 0$; that ${}^{\backprime}\Delta 3^2 = 5$, ${}^{\backprime}\Delta^2 3^2 = 2$, ${}^{\backprime}\Delta^3 3^2 = 0$; that $\Lambda\left(\frac{3}{2}\right)^2 = 3$, $\Lambda^3\left(\frac{3}{2}\right)^2 = 0$; and that $\mathrm{I}\Lambda 2^2 = 4$, $\mathrm{I}\Lambda^3 2^2 = 0$. Then applying the four series in question, respectively, we find

$$\mathrm{D}(-1)^2 = -1 - \frac{2}{2} = -2, \tag{257}$$

$$\mathrm{D}3^2 = 5 + \frac{2}{2} = 6, \tag{258}$$

$$\mathrm{D}\left(\frac{3}{2}\right)^2 = 3, \tag{259}$$

$$\mathrm{D}2^2 = 4. \tag{260}$$

This use of differences is especially important when, for any reason, it is desired to find the differentiate of a function of which certain arithmetical values are ascertained, but of which the law is unknown.

76. Besides such customary uses, these series, and particularly those expressed in terms of Δ and $'\Delta$, will, I think, be found of great value towards the elementary explanation of $\mathrm{D} = \log \mathrm{E}$. No student, informed that a differentiate is a series of differences, can fail to understand the statement. It is requisite to introduce the idea of infinity in some form, and these series will be found at least as intelligible as the vanishing fractions, and worthy of a place beside them in explanatory statements; essentially necessary, indeed, to complete a comprehensive view of the subject. It is certainly as easy to understand $\mathrm{D}x^2 = \Delta x^2 - \frac{1}{2}\Delta^2 x^2 = 2x$ as $\mathrm{D}x^2 = \frac{(x+h)^2 - x^2}{h}{}_{[h=0]} = 2x$. I would call attention to the fact that (253) supplies a verbal definition of a differentiate which may readily be borne in mind, namely, *the sum of divided lower differences.**

77. The application of these series to given forms of function will afford useful exercise to the student. To a^x, for example, any of them may at once be applied in the manner already exhibited. As another example, let $\phi x = x^m$. By the binomial theorem,

$$\mathrm{D}_0 (x+0)^m = \mathrm{D}_0 x^m + \mathrm{D}_0 0 m x^{m-1} + \mathrm{D}_0 0^2 m \frac{m-1}{2} x^{m-2} + \ldots . \tag{261}$$

Now $\Delta_0 (x+0)^m = (x+1)^m - x^m = \Delta x^m$, and similarly for second and higher differences; hence

$$\mathrm{D}_0 (x+0)^m = \left(\Delta_0 - \frac{1}{2}\Delta_0^2 + \ldots\right)(x+0)^m = \left(\Delta - \frac{1}{2}\Delta^2 + \ldots\right)x^m = \mathrm{D}x^m. \tag{262}$$

Also, $\mathrm{D}_0 x^m = (\Delta_0 - \ldots)\, x^m = 0$, and $\mathrm{D}_0 0 = (\Delta_0 - \ldots)\, 0 = 1$. As regards $\mathrm{D}_0 0^2$, $\mathrm{D}_0 0^3$, &c., we have it proved algebraically (De Morgan, *Calculus*, p. 255) that $\frac{\Delta^k 0^r}{k} = \Delta^{k-1} 0^{r-1} + \Delta^k 0^{r-1}$; and hence, when $r > 1$, we find by successive substitution that

$$\mathrm{D}_0 0^r = \left(\Delta_0 - \frac{1}{2}\Delta_0^2 + \ldots\right) 0^r = 0. \tag{263}$$

* De Morgan gives, in the article *Differential Calculus* of the *Penny Cyclopœdia*, certain comparative illustrations of the current definitions of $\mathrm{D}x^3$, according to the systems of Infinitesimals, Prime and Ultimate Ratios, Fluxions, and Limits, and the Residual Analysis of Landen. Each of these definitions requires several lines of print, the nearest approach to an equation being

$$\mathrm{D}x^3 = \frac{x^3 - x^3}{x - x} = 3x^2 .$$

Substituting these several results in (261), we have, finally,

$$\mathrm{D}x^m = mx^{m-1}. \tag{264}$$

78. We have now passed in review the more important equivalents for $\mathrm{D} = \log \mathrm{E}$ which may be deduced from the general differentiate-expression (220). We have seen how that theorem includes not only the known series in terms of Δ, $`\Delta$, and Λ, but also series in terms of an infinite number of other kinds of differences, besides the series in terms of DE^c; and how, in addition to such series, it comprehends an unlimited number of symbolic vanishing fractions, equivalents of D, one of which fractions exhibits, when applied in practice, the hitherto customary process of differentiation. Wide as that expression is, we shall shortly see that it is but a special case of a more comprehensive formula for the transformation of D^n; and, still further, we shall find that this latter formula is itself merely one case of a broader proposition regarding ∂^n, which again is but a special case of a general theorem relating to all functions of E. For the due presentation of this general theorem I find it necessary to lay down a new theory of factorials.

VI. *Theory of Factorials.*

79. Let

$$x^{[m]} = h^{m-1}x\,\frac{\Gamma(xh^{-1} + am)}{\Gamma(xh^{-1} + am - m + 1)}, \tag{265}$$

where a and h have the same value as in ∂, the difference-ratio symbol.* Whenever it is necessary to consider at the same time expressions involving more than one value of a or h, I shall add accents. Thus, while ∂ and $x^{[m]}$ are functions of a and h, ∂' and $x^{[m]'}$ will be the same functions of a' and h', and ∂'' and $x^{[m]''}$ those of a'' and h'', where a' and h', a'' and h'', may or may not be the same as a and h. Let us call (265) the general form of Primary Factorials; $x^{[3]}$, for example, being the general form of the primary factorial of the third degree.

80. By performing the operation, we find that

$$\partial x^{[m]} = mx^{[m-1]}. \tag{266}$$

81. Let us, except when otherwise expressly stated, consider only those cases in which m is neither negative nor fractional. We find that $x^{[0]} = 1$, that $x^{[1]} = x$, and that for all values of m greater than 1,

$$x^{[m]} = x\,(x + amh - h)(x + amh - 2h)\ \ldots\ (x + [a-1]\,mh + h), \tag{267}$$

* Here also (see note to paragraph 65) more symmetrical expressions can be had by writing $\frac{1}{2}(1-b)$ for a.

where a and h may have any value, positive or negative, whole or fractional. For example, let $a=17$ and $h=-\frac{1}{3}$; then

$$x^{[4]}=x\left(x-22\tfrac{1}{3}\right)(x-22)\left(x-21\tfrac{2}{3}\right). \tag{268}$$

When $a=0$, we have as a special case those functions to which, for positive indices, the name factorial has heretofore been confined.

82. When $h=0$, and a is finite, we have from (267)

$$x^{[m]}=x^m. \tag{269}$$

When $h=0$, and $a=-\frac{c}{h}$, where c is a finite quantity other than 0, we derive a remarkable function,

$$x^{[m]}=x\,(x-cm)^{m-1}. \tag{270}$$

When h is any quantity except 0 and 1, the expression $x^{[m]}$ may be reduced, by a suitable alteration of the variable, to a factorial form in which $h=1$. I shall, therefore, for brevity, omit the consideration of such values of h. The chief special cases when $h=1$ are those where $a=0$, $a=1$, and $a=\frac{1}{2}$, respectively,

$$x^{[m]}=x\,(x-1)(x-2)\ldots(x-m+1), \tag{271}$$

$$x^{[m]}=x\,(x+m-1)(x+m-2)\ldots(x+1), \tag{272}$$

$$x^{[m]}=x\left(x+\tfrac{1}{2}m-1\right)\left(x+\tfrac{1}{2}m-2\right)\ldots\left(x-\tfrac{1}{2}m+1\right). \tag{273}$$

The last form is no doubt novel. We may call these three varieties of factorials upper, lower, and central, respectively, to correspond with the analogous difference operations, Δ, $`\Delta$, and $\Lambda$; and I would suggest for them the special symbols $x^{m)}$, $x^{(m}$, and $x^{(m)}$, respectively. It is scarcely necessary to say that whatever may be proved true in general of $\partial$ and $x^{[m]}$ will hold good of D and $x^m$, $\Delta$ and $x^{m)}$, $`\Delta$ and $x^{(m}$, Λ and $x^{(m)}$, DE^c and $x\,(x-cm)^{m-1}$, as well as all other possible special cases.

83. By repetition of (266),

$$\partial^n x^{[m]}=m^{n)}x^{[m-n]}=\frac{\Gamma(1+m)}{\Gamma(1+m-n)}\,x^{[m-n]}. \tag{274}$$

Operating on both sides by ∂^{-n}, dividing by $m^{n)}$, and writing $m+n$ for m, we have

$$\partial^{-n}x^m=\frac{\Gamma(1+m)}{\Gamma(1+m+n)}\,x^{[m+n]}, \tag{275}$$

showing that (274) is true, as a principal form, when n is negative. If $x=0$, and $n<m$,

$$\partial^n 0^{[m]}=0; \tag{276}$$

and when $n = m$,

$$\partial^m 0^{[m]} = \Gamma(1+m). \tag{277}$$

Operating on this with ∂^{n-m}, supposing $n > m$, we have

$$\partial^n 0^{[m]} = 0, \tag{278}$$

because ∂ (*constant*) $= 0$. When, therefore, n is either greater or less than m, both being integral,

$$\partial^n 0^{[m]} = 0. \tag{279}$$

If ϕx can be expressed in positive integral powers, one of the terms being $a_m x^m$,

$$\phi\partial 0^{[m]} = a_m \partial^m 0^{[m]} = a_m \Gamma(1+m) = a_m \partial'^m 0^{[m]\prime} = \phi\partial' 0^{[m]\prime}. \tag{280}$$

Again,

$$\phi(c\partial)\, 0^{[m]} = c^m a_m \partial^m 0^{[m]} = c^m \phi\partial 0^{[m]} = c^m \phi\partial' 0^{[m]\prime}. \tag{281}$$

84. Any factorial $x^{[m]}$ may be expressed in factorials of any other form, such as $x^{[m]\prime}$; for $x^{[m]}$ is by definition, let us say, $x^m + c_m x^{m-1} + \ldots$, and $x^{[m]\prime}$ is similarly $x^m + c_m' x^{m-1} + \ldots$, wherefore $x^{[m]}$, an algebraic expression of the mth degree, may be replaced by $x^{[m]\prime}$, also of the mth degree, plus factorials of lower degrees, the coefficients of which may be determined from the data, which are sufficient for that purpose. In general, therefore, when $n > m$,

$$\partial'^n 0^{[m]} = \partial'^n (0^{[m]\prime} + \ldots) = 0, \tag{282}$$

and when $n = m$,

$$\partial'^m 0^{[m]} = \partial'^m 0^{[m]\prime} = \Gamma(1+m). \tag{283}$$

85. Again,

$$\phi\partial 0^{[m]} = a_m \partial^m 0^{[m]} = a_m \partial^{m-1} \partial 0^{[m]} = a_m m \partial^{m-1} 0^{[m-1]} = \phi' \partial 0^{[m-1]}, \tag{284}$$

and by repetition,

$$\phi\partial 0^{[m]} = \phi^n \partial 0^{[m-n]} = \phi^n \partial' 0^{[m-n]\prime}, \tag{285}$$

where $\phi^n x = \mathrm{D}^n \phi x$. For example,

$$\varepsilon^{h\partial} 0^{[m]} = h^m \varepsilon^{h\partial} 0^{[0]} = h^m (1 + h\partial + \ldots)\, 0^{[0]} = h^m. \tag{286}$$

It follows from (285) and (152) that, when $\phi\partial$ can be expressed in positive integral powers of ∂,

$$\phi\partial 0^{[m]} = \phi^m \mathrm{D} 0^0 = \phi^m 0 = \mathrm{D}^m \phi 0. \tag{287}$$

86. Suppose

$$\phi\mathrm{E} = \psi\partial = a_0 + a_1\partial + a_2\partial^2 + \ldots. \tag{288}$$

By Herschel's theorem (162),

$$\psi\partial = \psi\mathrm{D}0^0 + \psi\mathrm{D}0 \,.\, \partial + \frac{1}{2}\psi\mathrm{D}0^2 \,.\, \partial^2 + \ldots, \tag{289}$$

and by Maclaurin's theorem (120),

$$\psi\partial = \mathrm{D}^0\psi 0 + \mathrm{D}\psi 0 \,.\, \partial + \frac{1}{2}\mathrm{D}^2\psi 0 \,.\, \partial^2 + \ldots. \tag{290}$$

From (288) we have at once, since $\phi\mathrm{E}0^{[m]} = a_m \partial^m 0^{[m]} = 2\,.\,3\,.\,4 \ldots m a_m$, and therefore $a_m = \dfrac{1}{2\,.\,3\,.\,4 \ldots m}\, \phi\mathrm{E}0^{[m]}$,

$$\phi\text{E} = \phi\text{E}0^{[0]} + \phi\text{E}0^{[1]} . \partial + \frac{1}{2}\, \phi\text{E}0^{[2]} . \partial^2 + \frac{1}{2 . 3}\, \phi\text{E}0^{[3]} . \partial^3 + \dots . \qquad (291)$$

The same result may be had from (289) or (290), observing respectively (280) or (287). This is the general theorem referred to in paragraph 78. It may fitly be named *the factorial theorem.* It includes as special cases a very large number of known propositions of the Differential Calculus and Calculus of Finite Differences, and affords a ready instrument for the discovery of relations hitherto unnoticed. It should be laid down among the earliest propositions in any formal treatise on the Calculus. Recurring to the line of thought pursued in paragraph 52, we may remark that this theorem applies in all cases where ϕE produces determinate results, since it holds good, as proved, for all functions of E which can be expressed in positive integral powers of ∂. The following is a variation which results from (281):

$$\psi\,(c\partial) = \psi\partial' 0^{[0]'} + \psi\partial' 0^{[1]'} . c\partial + \frac{1}{2}\, \psi\partial' 0^{[2]'} . c^2\partial^2 + \dots . \qquad (292)$$

Here ∂' and $x^{[m]'}$ may or may not be the same as ∂ and $x^{[m]}$, and c may have any value. Of course, $\phi\text{E}0^{[0]} = \phi 1$, and $\phi\partial 0^{[0]} = \phi 0$.

87. An important case of the factorial theorem is that where $\phi\text{E} = \text{E}^k$. Applying it to ψx, we have the following *generalization of Taylor's theorem:*

$$\psi\,(x+k) = \psi x + k\partial\psi x + \frac{1}{2}\, k^{[2]}\partial^2\psi x + \dots . \qquad (293)$$

If $\psi x = x^{[m]'}$, we obtain a *generalization of the binomial theorem*, true for all values of m, including negative and fractional values,

$$(x+k)^{[m]'} = x^{[m]'} + k\partial x^{[m]'} + \frac{1}{2}\, k^{[2]}\partial^2 x^{[m]'} + \dots . \qquad (294)$$

This enables us to expand any binomial factorial in factorials of any other desired form. For example, to expand $(x+k)^{[m]}$ in factorials of k, we have this minor generalization of the binomial theorem,

$$(x+k)^{[m]} = x^{[m]} + mx^{[m-1]}k + m\,\frac{m-1}{2}\, x^{[m-2]}k^{[2]} + \dots , \qquad (295)$$

good for all values of m; or, to expand $(x+k)^m$ in general primary factorials,

$$(x+k)^m = x^m + k\partial x^m + \frac{1}{2}\, k^{[2]}\partial^2 x^m + \dots . \qquad (296)$$

If $x = 0$, we have from (294), for the expansion of a factorial in factorials of any other kind, whether m be positive or negative, whole or fractional,

$$k^{[m]'} = k\partial 0^{[m]'} + \frac{1}{2}\, k^{[2]}\partial^2 0^{[m]'} + \dots . \qquad (297)$$

If, in (293), $x = 0$, we have a *generalization of Maclaurin's theorem,*

$$\psi k = \psi 0 + k\partial\psi 0 + \frac{1}{2}\, k^{[2]}\partial^2\psi 0 + \dots . \qquad (298)$$

These various theorems for factorial expansion will be found capable of many useful and interesting applications. For example, in (298) let $\psi k = c^k$, and let us write x for k; then

$$c^x = 1 + x\partial_0 c^0 + \frac{1}{2} x^{[2]}\partial_0^2 c^0 + \ldots, \tag{299}$$

a *generalization of the exponential theorem.* An expression for the value of log $(1+x)$ may similarly be derived, of which one case is this known series,

$$\log(1+x) = \log 2 . x + \frac{1}{2} \log \frac{3}{2^2} . x(x-1) + \ldots. \tag{300}$$

If, in (299), we write ε for c and $k\mathrm{D}$ for x, we have this result,

$$\mathrm{E}^k = 1 + \partial_0\varepsilon^0 . k\mathrm{D} + \ldots, \tag{301}$$

$$\phi(x+k) = \phi x + \partial_0\varepsilon^0 . k\mathrm{D}\phi x + \frac{1}{2}\partial_0^2\varepsilon^0 . (k\mathrm{D})^{[2]}\phi x + \ldots. \tag{302}$$

88. A certain variation of (299) is so remarkable as to be worthy of extended notice. Let g be such that $\partial g^x = g^x$, that is to say, that

$$\frac{g^{x+(1-a)h} - g^{x-ah}}{h} = g^x; \tag{303}$$

then

$$\frac{g^{(1-a)h} - g^{-ah}}{h} = 1, \tag{304}$$

whence

$$g^h - 1 = hg^{ah}. \tag{305}$$

The solution of one of these equations will give the value of g. If more than one solution presents itself, that only can be accepted which agrees with the condition $\partial g^x = g^x$. When $h = 0$, equation (304) becomes log $g = 1$, or $g = \varepsilon$. From (299) we have at once the series in question,

$$g^x = 1 + x + \frac{1}{2} x^{[2]} + \frac{1}{2.3} x^{[3]} + \ldots, \tag{306}$$

of which the exponential theorem (23) is a special case. If $h = 1$ and $a = 0$, we have a series verifiable by expanding $(1+1)^x$,

$$2^x = 1 + x + \frac{1}{2} x^{2)} + \ldots. \tag{307}$$

If $h = 1$ and $a = \frac{1}{2}$,

$$\left(\frac{3+\sqrt{5}}{2}\right)^x = 1 + x + \frac{1}{2} x^{(2)} + \ldots. \tag{308}$$

If $h = 2$ and $a = \frac{1}{2}$,

$$(1+\sqrt{2})^x = 1 + x + \frac{1}{2} x^{[2]} + \ldots, \tag{309}$$

where $x^{[2]}=x^2$, $x^{[3]}=x(x+1)(x-1)$, $x^{[4]}=x(x+2)\,x\,(x-2)$, $x^{[5]}=x(x+3)(x+1)(x-1)(x-3)$, and so on. If $h=\frac{1}{2}$ and $a=0$,

$$\left(\frac{9}{4}\right)^x=1+x+\frac{1}{2}x^{[2]}+\ldots, \tag{310}$$

where $x^{[2]}=x\left(x+\frac{1}{2}\right)$, $x^{[3]}=x\left(x-\frac{1}{2}\right)(x-1)$, and so on. If $h=\frac{1}{2}$ and $a=1$, similarly,

$$4^x=1+x+\frac{1}{2}x^{[2]}+\ldots, \tag{311}$$

where $x^{[2]}=x\left(x+\frac{1}{2}\right)$, $x^{[3]}=x\left(x+\frac{1}{2}\right)(x+1)$, and so on. It is needless to multiply these illustrations, which show that ε is but a type of an infinite number of constants, and that the exponential theorem is but the corresponding type of an infinite number of factorial series of the same general form. The series above given may be verified, of course, by arithmetical approximation. It may be shown further that ε^x itself is capable of an unlimited number of factorial expansions, all having the same coefficients as the exponential theorem. In general,

$$\varepsilon^x=1+x+\frac{1}{2}x^{[2]}+\frac{1}{2\,.\,3}x^{[3]}+\ldots, \tag{312}$$

where $a=h^{-1}\log\frac{\varepsilon^h-1}{h}$. Thus, if $h=1$, $a=\log(\varepsilon-1)$, and $x^{[2]}=x(x+2a-1)$, $x^{[3]}=x(x+3a-1)(x+3a-2)$, and so on; if $h=\log(1+h)=0.01$ nearly, $a=0$, and $x^{[2]}=x(x-h)$, $x^{[3]}=x(x-h)(x-2h)$, and so on; and if $h=\log(1+h\varepsilon)=1.75$ nearly, $a=h^{-1}$, and $x^{[2]}=x(x+2-h)$, $x^{[3]}=x(x+3-h)(x+3-2h)$, and so on.

89. Let z be that function of x which ∂ is of E, namely, $\frac{x^{(1-a)h}-x^{-ah}}{h}$, so that $x=(1+hzx^{ah})^{\frac{1}{h}}$. Since $\phi\text{E}_0x^0=\phi x$, $\partial_0^n x^0=z^n$. Applying the factorial theorem to x^0, we obtain a *generalization of Herschel's theorem*,

$$\phi x=\phi\text{E}0^0+z\phi\text{E}0+\frac{1}{2}z^2\phi\text{E}0^{[2]}+\ldots. \tag{313}$$

If $\phi x=\psi z$, $\phi\text{E}=\psi\partial$. If $h=0$, $z=\log x$, $x=\varepsilon^z$, $0^{[n]}=0^n$, and we have, as a special case, Herschel's theorem. If $h=1$ and $a=0$, $x=1+z$, and

$$(1+z)=\phi\text{E}0^0+z\phi\text{E}0+\frac{1}{2}z^2\phi\text{E}0^{2)}+\ldots; \tag{314}$$

while if $h=1$ and $a=1$, $x=\frac{1}{1-z}$, and

$$\phi\frac{1}{1-z}=\phi\text{E}0^0+z\phi\text{E}0+\frac{1}{2}z^2\phi\text{E}0^{(2}+\ldots. \tag{315}$$

To illustrate these apparently novel expansion-theorems, take

$$(1+z)^n = \text{E}^n0^0 + z\text{E}^n0 + \frac{1}{2}z^2\text{E}^n0^{2)} + \ldots$$

$$= 1 + zn + \frac{1}{2}z^2n^{2)} + \ldots, \tag{316}$$

$$\left(\frac{1}{1-z}\right)^n = 1 + zn + \frac{1}{2}z^2n^{(2} + \ldots, \tag{317}$$

as by the binomial theorem. By putting $h = zx$, we easily derive from (314) and (315) two *variations of Taylor's theorem:*

$$\phi\,(x+h) = \phi\,(x\text{E})0^0 + hx^{-1}\phi\,(x\text{E})0 + \frac{1}{2}h^2x^{-2}\phi\,(x\text{E})0^{2)} + \ldots, \tag{318}$$

$$\phi\,(x+h) = \phi\,\frac{x}{\text{E}}\,0^0 - hx^{-1}\phi\,\frac{x}{\text{E}}\,0 + \frac{1}{2}h^2x^{-2}\phi\,\frac{x}{\text{E}}\,0^{(2} - \ldots. \tag{319}$$

These formulæ are particularly worthy of attention. In them, we have Taylor's theorem demonstrated, and the coefficients expressed, without any reference whatever to the operation of differentiation; a result, the possibility of which would have seemed incredible. We obtain by this means expressions equivalent to $\text{D}\phi x$ which may be added, with advantage, to the list of those already considered; the chief advantage being that they are neither vanishing fractions nor infinite series, and cannot, indeed, be looked upon as in any respect transcendental. They are,

$$\text{D}\phi x = x^{-1}\,(x\text{E})\,0, \tag{320}$$

$$\text{D}\phi x = -\,x^{-1}\phi\,\frac{x}{\text{E}}\,0. \tag{321}$$

For example,

$$\text{D}\varepsilon^x = x^{-1}\varepsilon^{x+x\Delta}0 = x^{-1}\varepsilon^x\,(1 + x\Delta + \ldots)\,0 = \varepsilon^x, \tag{322}$$

$$\text{D}\log x = x^{-1}\,(\log x + \log\,[1+\Delta])\,0 = x^{-1}\left(\Delta - \frac{1}{2}\Delta^2 + \ldots\right)0 = x^{-1}, \tag{323}$$

$$\text{D}x^m = x^{-1}x^m\text{E}^m0 = mx^{m-1}. \tag{324}$$

These expressions for $\text{D}\phi x$ may be proved independently of Taylor's theorem. Thus, if a_mx^m be the general term of ϕx, that of $\text{D}\phi x$ will be a_mmx^{m-1}, which is also that of $x^{-1}\phi\,(x\text{E})\,0$, or $x^{-1}a_mx^m\text{E}^m0$. In general, similarly,

$$\text{D}^n\phi x = x^{-n}\phi\,(x\text{E})\,0^{n)}, \tag{325}$$

$$\text{D}^n\phi x = (-1)^nx^{-n}\phi\,\frac{x}{\text{E}}\,0^{(n}. \tag{326}$$

For example, n being a positive integer,

$$\text{D}^n\log x = x^{-n}\,(\log x + \log\,[1+\Delta])\,0^{n)} = x^{-n}\,(-1)^{n-1}\frac{1}{n}\Delta^n0^{n)}.\text{*} \tag{237}$$

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* Since this was written, I have found other forms of $\text{D}^n\phi x$, and therefore of Taylor's theorem. By (110), $\phi\text{E}_0c^0 = \phi c$; hence, $\phi\,(x+\Delta_0)(1+h)^0 = \phi\,(x+h)$, and

$$\phi\,(x+h) = \phi x + h\phi\,(x+\Delta)\,0 + \frac{1}{2}h^2\phi\,(x+\Delta)\,0^{2)} + \ldots.$$

90. If, in (313), we write $\phi \log$ for ϕ, we have a logarithmic formula of extraordinary generality, which may properly be called *the logarithmic theorem*:

$$\phi \log x = \phi \mathrm{D} 0^0 + z\phi \mathrm{D} 0 + \frac{1}{2} z^2 \phi \mathrm{D} 0^{[2]} + \ldots \tag{328}$$

An obviously important case is that where $\phi \log x = (\log x)^n$, and of this, again, the most common and most useful special case is

$$\log x = z + \frac{1}{2} z^2 \mathrm{D} 0^{[2]} + \frac{1}{2 \,.\, 3} z^3 \mathrm{D} 0^{[3]} + \ldots \tag{329}$$

We have thus, though in a different and more perspicuous shape, the general logarithmic series (31). To reduce it to the algebraic form of (31), we have only to put $y = hz$, and to observe that

$$\mathrm{D} 0^{[2]} = \mathrm{D}_0 0\, (0 + 2ah - h) = 2ah - h\,, \tag{330}$$

and so on, the general rule being

$$\mathrm{D} 0^{[n]} = \mathrm{D} 0 \phi 0 = 0 \mathrm{D} \phi 0 + \phi 0 \mathrm{D} 0 = \phi 0\,. \tag{331}$$

91. Of the various functions of E which may be expressed by the aid of the factorial theorem in terms of ∂, the most important are those other difference-ratios, of whatever degree, collectively represented by the symbol ∂'^n. When $\phi \mathrm{E} = \partial'^n$, all terms of the expansion prior to that containing $\partial'^n 0^{[n]}$ vanish, as may be seen from (282), and we have remaining the following *difference-ratio transformation formula:*

$$\partial'^n = \partial^n + \frac{\partial'^n 0^{[n+1]}}{\Gamma(n+2)} \partial^{n+1} + \frac{\partial'^n 0^{[n+2]}}{\Gamma(n+3)} \partial^{n+2} + \ldots \tag{332}$$

Of this formula one or two special cases are already known, as where ∂ and ∂' are respectively D and Δ, and *vice versa;* though it does not appear to be admitted, in the discussion of those cases, that any negative value can be assigned to n.* I shall now show that the general formula, including, of course, the cases just referred to, holds good when n is negative. That $\partial \phi^{[0]} = 0 \phi^{[-1]} = \frac{\Gamma 1}{\Gamma 0} x^{[-1]}$ follows from (274). If it be doubted, we may, for present purposes, define $\Gamma 0$ to be $\Gamma 1 \frac{x^{[-1]}}{\partial x^{[0]}}$, $\Gamma(-1)$ to be $\frac{\Gamma 0}{-1}$, and so on, so that (274) may hold good for all possible integral values of m and n. The quantities so defined are imaginary, indeed, but if their use enables us to reach

Here $\mathrm{D}^n \phi x = \phi(x + \Delta) 0^{n)}$. Similarly, $\phi(x + `\Delta_0)(1 - h)^{-0} = \phi(x + h)$, and $\mathrm{D}^n \phi x = \phi(x + `\Delta)\, 0^{(n}$. In general, $\mathrm{D}^n \phi x = \phi(x + \partial_0)\, 0^{[n]}$, and, still more generally, $h^n \mathrm{D}^n \phi(kx) = k^n \phi(kx + h\partial_0)\, 0^{[n]}$; an expression readily derived from (167) and (280). If $n = 1$, we have $\mathrm{D}\phi x = \phi(x + \partial_0)\, 0$. For example, $\mathrm{D}\varepsilon^{x^2} = \varepsilon^{x^2} \varepsilon^{2x\partial + \partial}\, 0 = \varepsilon^{x^2}(1 + 2x\partial + \ldots)\, 0 = \varepsilon^{x^2} 2x$.

* Compare Boole, *Finite Differences*, 2d ed., p. 24.

finite results, no valid objection can be made to it. We find by the factorial theorem, n being supposed negative, that

$$\left(\frac{\partial}{\partial'}\right)^{-n} = \left(\frac{\partial}{\partial'}\right)^{-n} 0^0 + \left(\frac{\partial}{\partial'}\right)^{-n} 0 \,.\, \partial + \frac{1}{2}\left(\frac{\partial}{\partial'}\right)^{-n} 0^{[2]} \,.\, \partial^2 + \dots \tag{333}$$

From (274),

$$\left(\frac{\partial}{\partial'}\right)^{-n} 0^{[m]} = \partial'^{n} 0^{[m+n]} \frac{\Gamma(1+m)}{\Gamma(1+m+n)} . \tag{334}$$

Making the proper substitutions, and multiplying both sides of (333) by ∂^n, we have (332) true when n is negative. That the coefficients are in that case finite may be seen, for

$$\left(\frac{\partial}{\partial'}\right)^{-n} 0^{[m]} = \partial^{-n} \partial'^{n} 0^{[m]}, \tag{335}$$

and since $0^{[m]}$ may be expressed in factorials of the form $0^{[m]'}$, neither the operation ∂'^{n} nor the subsequent operation ∂^{-n} can destroy its finite character.

92. The coefficient of ∂^{n+r} in (332) is $\dfrac{\partial'^{n} 0^{[n+r]}}{\Gamma(n+r+1)}$. This class of coefficients is likely to become so important that it will be desirable to assign to it a special symbol, by way of abbreviation. Let the general symbol be ${}^{[r]}\partial'^{n}$, and let the same device be employed in all special cases, so that, for example,

$$\frac{\mathrm{D}^n 0^{(n+r)}}{\Gamma(n+r+1)} = {}^{(r)}\mathrm{D}^n, \tag{336}$$

$$\frac{\Delta^n 0^{n+r}}{\Gamma(n+r+1)} = {}^{r}\Lambda^n, \tag{337}$$

and so on. We may, therefore, wherever we see an index prefixed to a difference-ratio symbol, understand that it indicates a constant coefficient.

93. Using these symbols, we may thus write the difference-ratio transformation formula (332):

$$\partial'^{n} = \partial^n + {}^{[1]}\partial'^{n} \,.\, \partial^{n+1} + {}^{[2]}\partial'^{n} \,.\, \partial^{n+2} + \dots \tag{338}$$

If $\partial' = \mathrm{D}$,

$$\mathrm{D}^n = \partial^n + {}^{[1]}\mathrm{D}^n \,.\, \partial^{n+1} + {}^{[2]}\mathrm{D}^n \,.\, \partial^{n+2} + \dots, \tag{339}$$

where, if $n = 1$,

$$\mathrm{D} = \partial + {}^{[1]}\mathrm{D}^1 \,.\, \partial^2 + {}^{[2]}\mathrm{D}^1 \,.\, \partial^3 + \dots, \tag{340}$$

which is merely a concise way of writing the general differentiate expression (220). The interpretation of ${}^{[m]}\mathrm{D}^1$ has been illustrated in paragraph 90, in discussing the corresponding logarithmic series.

94. Perhaps the next most important special case of the transformation formula (332, 338) is that where $n = -1$,

$$\partial'^{-1} = \partial^{-1} + {}^{[1]}\partial'^{-1} \,.\, \partial^0 + {}^{[2]}\partial'^{-1} \,.\, \partial + \dots, \tag{341}$$

a most comprehensive *summation formula*, which includes as many special cases as there are ways of combining the special forms of ∂, such as D, Δ, $`\Delta$, Λ, &c., including, as we shall shortly see, those mean central differences whose symbol is $\mathrm{I}\Delta^n$. Among these special cases, which, having been thus distinctly indicated, it is needless to write out at present, are several known formulæ; known, that is to say, in substance, though the correct form of their coefficients is probably a novelty. One of these is the celebrated series of Maclaurin and Euler, namely, in our notation,

$$\Delta^{-1} = \mathrm{D}^{-1} + {}^1\Delta^{-1}.\mathrm{D}^0 + {}^2\Delta^{-1}.\mathrm{D} + \ldots, \tag{342}$$

the coefficients of which are factors of the formerly inexplicable Numbers of Bernoulli.* We may write (341) in this form,

$$\partial'^{-1} = (1 + {}^{[1]}\partial'^{-1}.\partial + {}^{[2]}\partial'^{-1}.\partial^2 + \ldots)\,\partial^{-1}, \tag{343}$$

whence

$$\partial'^{-1}\partial = 1 + {}^{[1]}\partial'^{-1}.\partial + {}^{[2]}\partial'^{-1}.\partial^2 + \ldots. \tag{344}$$

Various special cases of these two formulæ will be found useful in practice. When $\partial = \Delta$ and $\partial' = \mathrm{D}$, we derive from (344) the well-known formula of Laplace customarily employed for mechanical quadrature.

95. The coefficients required in the more important applications of the factorial theorem ought to be tabulated. This is particularly true of those coefficients, represented by ${}^{[r]}\partial^n$, which are needed in applying the difference-ratio transformation formula (338), a theorem of which many cases will become increasingly important in the future. As a specimen of what should be done in the tabulation of coefficients, I give now a table of ${}^r\Delta^n$ and ${}^{r)}\mathrm{D}^n$, computed with due care, which will be found useful in applying these two formulæ,

$$\Delta^n = \mathrm{D}^n + {}^1\Delta^n.\mathrm{D}^{n+1} + {}^2\Delta^n.\mathrm{D}^{n+2} + \ldots, \tag{345}$$

and

$$\mathrm{D}^n = \Delta^n + {}^{1)}\mathrm{D}^n.\Delta^{n+1} + {}^{2)}\mathrm{D}^n.\Delta^{n+2} + \ldots. \tag{346}$$

These formulæ are deduced from (338) by putting $\partial = \mathrm{D}$ and $\partial' = \Delta$, and *vice versa.* They are well known for positive values of n, though the coefficients do not seem to have been tabulated; and are true, as already shown, for negative values. The table contains, of course, the coefficients of x^{n+r} in $(\varepsilon^x - 1)^n$ and $[\log(1+x)]^n$. When $r < 0$, ${}^r\Delta^n = {}^{r)}\mathrm{D}^n = 0$. [See Note, pp. 150 and 151.]

* These numbers are, ${}^1\Delta^{-1} = -\frac{1}{2}$, $2.{}^2\Delta^{-1} = 2\Delta^{-1}0 = \frac{1}{6}$, $2.3.4.{}^4\Delta^{-1} = 4\Delta^{-1}0^3 = -\frac{1}{30}$, $2.3.4.5.6.{}^6\Delta^{-1} = 6\Delta^{-1}0^5 = \frac{1}{42}$, and so on.

As a mere object of curiosity, this table is very remarkable. An unlimited number of general formulæ exhibiting relations between the tabular numbers may be devised. Some fifty which I have noticed, including a small number derived from known expressions by dividing by $\Gamma(n+r+1)$, are appended. Negative values may be assigned to c.

Equivalents of ${}^{r}\Delta^{n}$:

$${}^{r)}\mathrm{D}^{-n-r-1} - {}^{r-1}\Delta^{n+1},$$

$$\frac{n+r+1}{n+1}.{}^{r}\Delta^{n+1} - {}^{r-1}\Delta^{n+1},$$

$$-{}^{1)}\mathrm{D}^{n}.{}^{r-1}\Delta^{n+1} - {}^{2)}\mathrm{D}^{n}.{}^{r-2}\Delta^{n+2} - \ldots,$$

$$-\frac{n}{n+1}.{}^{1}\Delta^{-n-1}.{}^{r-1}\Delta^{n+1} - \frac{n}{n+2}.{}^{2}\Delta^{-n-2}.{}^{r-2}\Delta^{n+2} - \ldots,$$

$$\frac{1}{2^{r}-1}\left(n\,\frac{{}^{r-1}\Delta^{n+1}}{2} + n\,\frac{n-1}{2}\,\frac{{}^{r-2}\Delta^{n+2}}{2^{2}} + n\,\frac{n-1}{2}\,\frac{n-2}{3}\,\frac{{}^{r-3}\Delta^{n+3}}{2^{3}} + \ldots\right),$$

$$(n+r+1)\left(\frac{{}^{r}\Delta^{n+1}}{n+1} - \frac{{}^{r-1}\Delta^{n+2}}{n+2} + \frac{{}^{r-2}\Delta^{n+3}}{n+3} - \ldots\right),$$

$${}^{r}\Delta^{n+c} + {}^{1)}\mathrm{D}^{c}.{}^{r-1}\Delta^{n+c+1} + {}^{2)}\mathrm{D}^{c}.{}^{r-2}\Delta^{n+c+2} + \ldots,$$

$${}^{r}\Delta^{n+c} + \frac{c}{c+1}.{}^{1}\Delta^{-c-1}.{}^{r-1}\Delta^{n+c+1} + \frac{c}{c+2}.{}^{2}\Delta^{-c-2}.{}^{r-2}\Delta^{n+c+2} + \ldots,$$

$$-{}^{1)}\mathrm{D}^{n+r-1}.{}^{r-1}\Delta^{n} - {}^{2)}\mathrm{D}^{n+r-2}.{}^{r-2}\Delta^{n} - \ldots,$$

$$-\frac{n+r-1}{n+r}.{}^{1}\Delta^{-n-r}.{}^{r-1}\Delta^{n} - \frac{n+r-2}{n+r}.{}^{2}\Delta^{-n-r}.{}^{r-2}\Delta^{n} - \ldots,$$

$$\frac{n}{n+r}\left({}^{r}\Delta^{n-1} + {}^{r-1}\Delta^{n-1} + \frac{{}^{r-2}\Delta^{n-1}}{1.2} + \ldots\right),$$

$${}^{r}\Delta^{n+c} + {}^{1}\Delta^{-c}.{}^{r-1}\Delta^{n+c} + {}^{2}\Delta^{-c}.{}^{r-2}\Delta^{n+c} + \ldots,$$

$${}^{r}\Delta^{n+c} + \frac{c}{c-1}.{}^{1)}\mathrm{D}^{c-1}.{}^{r-1}\Delta^{n+c} + \frac{c}{c-2}.{}^{2)}\mathrm{D}^{c-2}.{}^{r-2}\Delta^{n+c} + \ldots,$$

$$r.{}^{r}\Delta^{n-c} - r\,\frac{r-1}{2}.{}^{r}\Delta^{n-2c} + \ldots + \left(\frac{c}{2}\right)^{r},$$

$$-\frac{n}{n+r-1}.{}^{1)}\mathrm{D}^{n+r-1}.{}^{r-1)}\mathrm{D}^{1-n-r} - \frac{n}{n+r-2}.{}^{2)}\mathrm{D}^{n+r-2}.{}^{r-2)}\mathrm{D}^{2-n-r} - \ldots,$$

$$-\frac{n}{n+r}\left({}^{1}\Delta^{-n-r}.{}^{r-1)}\mathrm{D}^{1-n-r} + {}^{2}\Delta^{-n-r}.{}^{r-2)}\mathrm{D}^{2-n-r} + \ldots\right),$$

$$n\,\frac{n-1}{n+r}\left(\frac{{}^{r)}\mathrm{D}^{1-n-r}}{n+r-1} + \frac{{}^{r-1)}\mathrm{D}^{2-n-r}}{n+r-2} + \frac{{}^{r-2)}\mathrm{D}^{3-n-r}}{(n+r-3)\,1.2} + \ldots\right),$$

$$\frac{c-n}{c-n-r}.{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n}{c-n-r+1}.{}^{1}\Delta^{c}.{}^{r-1)}\mathrm{D}^{c-n-r+1} + \ldots,$$

$$\frac{c-n}{c-n-r}.{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n}{c-n-r+1}\,\frac{c}{c+1}.{}^{1)}\mathrm{D}^{-c-1}.{}^{r-1)}\mathrm{D}^{c-n-r+1} + \ldots,$$

$$r\,\frac{c-n}{c-n-r}.{}^{r)}\mathrm{D}^{c-n-r} - r\,\frac{r-1}{2}\,\frac{2c-n}{2c-n-r}.{}^{r)}\mathrm{D}^{2c-n-r} + \ldots + \left(\frac{c}{2}\right)^{r},$$

$$-\frac{n}{n+r}\left({}^{1}\Delta^{-n-1}.{}^{r-1)}\mathrm{D}^{-n-r} + {}^{2}\Delta^{-n-2}.{}^{r-2)}\mathrm{D}^{-n-r} + \ldots\right),$$

$$-\frac{n+1}{n+r}.{}^{1)}\mathrm{D}^{n}.{}^{r-1)}\mathrm{D}^{-n-r} - \frac{n+2}{n+r}.{}^{2)}\mathrm{D}^{n}.{}^{r-2)}\mathrm{D}^{-n-r} - \ldots,$$

$$\frac{n}{n+r}\,\frac{1}{2^{r}-1}\left(\frac{n+1}{2}.{}^{r-1)}\mathrm{D}^{-n-r} + \frac{n+2}{2^{2}}\,\frac{n-1}{2}.{}^{r-2)}\mathrm{D}^{-n-r} + \frac{n+3}{2^{3}}\,\frac{n-1}{2}\,\frac{n-2}{3}.{}^{r-3)}\mathrm{D}^{-n-r} + \ldots\right),$$

$${}^{r)}\mathrm{D}^{-n-r-1} - {}^{r-1)}\mathrm{D}^{-n-r-1} + {}^{r-2)}\mathrm{D}^{-n-r-1} - \ldots,$$

$$\frac{c-n}{c-n-r}.{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n-1}{c-n-r}.{}^{1)}\mathrm{D}^{-c}.{}^{r-1)}\mathrm{D}^{c-n-r} + \ldots,$$

$$\frac{c-n}{c-n-r}.{}^{r)}\mathrm{D}^{c-n-r} + \frac{c-n-1}{c-n-r}\,\frac{c}{c-1}.{}^{1}\Delta^{c-1}.{}^{r-1)}\mathrm{D}^{c-n-r} + \ldots.$$

Equivalents of ${}^{r)}\mathrm{D}^{n}$:

${}^{r}\Delta^{-n-r-1} + {}^{r-1}\Delta^{-n-r}$,

$\frac{n+r+1}{n+1}\cdot{}^{r)}\mathrm{D}^{n+1} + \frac{n+r}{n+1}\cdot{}^{r-1)}\mathrm{D}^{n+1}$,

$-{}^{1}\Delta^{n}\cdot{}^{r-1)}\mathrm{D}^{n+1} - {}^{2}\Delta^{n}\cdot{}^{r-2)}\mathrm{D}^{n+2} - \dots$,

$-\frac{n}{n+1}\cdot{}^{1)}\mathrm{D}^{-n-1}\cdot{}^{r-1)}\mathrm{D}^{n+1} - \frac{n}{n+2}\cdot{}^{2)}\mathrm{D}^{-n-2}\cdot{}^{r-2)}\mathrm{D}^{n+2} - \dots$,

$\frac{-1}{2^r-1}\left(\frac{n+r-1}{2}\cdot{}^{r-1)}\mathrm{D}^{n} - \frac{n+r-2}{2^2}\,\frac{n+r+1}{2}\cdot{}^{r-2)}\mathrm{D}^{n} + \frac{n+r-3}{2^3}\,\frac{n+r+1}{2}\,\frac{n+r+2}{3}\cdot{}^{r-3)}\mathrm{D}^{n} - \dots\right)$,

$(n+r+1)\left(\frac{{}^{r)}\mathrm{D}^{n+1}}{n+1} + \frac{{}^{r-1)}\mathrm{D}^{n+2}}{n+2} + \frac{{}^{r-2)}\mathrm{D}^{n+3}}{(n+3)\,1.2} + \dots\right)$,

${}^{r)}\mathrm{D}^{n+c} + {}^{1}\Delta^{c}\cdot{}^{r-1)}\mathrm{D}^{n+c+1} + {}^{2}\Delta^{c}\cdot{}^{r-2)}\mathrm{D}^{n+c+2} + \dots$,

${}^{r)}\mathrm{D}^{n+c} + \frac{c}{c+1}\cdot{}^{1)}\mathrm{D}^{-c-1}\cdot{}^{r-1)}\mathrm{D}^{n+c+1} + \frac{c}{c+2}\cdot{}^{2)}\mathrm{D}^{-c-2}\cdot{}^{r-2)}\mathrm{D}^{n+c+2} + \dots$,

$-{}^{1}\Delta^{n+r-1}\cdot{}^{r-1)}\mathrm{D}^{n} - {}^{2}\Delta^{n+r-2}\cdot{}^{r-2)}\mathrm{D}^{n} - \dots$,

$-\frac{n+r-1}{n+r}\cdot{}^{1)}\mathrm{D}^{-n-r}\cdot{}^{r-1)}\mathrm{D}^{n} - \frac{n+r-2}{n+r}\cdot{}^{2)}\mathrm{D}^{-n-r}\cdot{}^{r-2)}\mathrm{D}^{n} - \dots$,

$\frac{n}{n+r}\left({}^{r)}\mathrm{D}^{n-1} - {}^{r-1)}\mathrm{D}^{n-1} + {}^{r-2)}\mathrm{D}^{n-1} - \dots\right)$,

${}^{r)}\mathrm{D}^{n+c} + {}^{1)}\mathrm{D}^{-c}\cdot{}^{r-1)}\mathrm{D}^{n+c} + {}^{2)}\mathrm{D}^{-c}\cdot{}^{r-2)}\mathrm{D}^{n+c} + \dots$,

${}^{r)}\mathrm{D}^{n+c} + \frac{c}{c-1}\cdot{}^{1}\Delta^{c-1}\cdot{}^{r-1)}\mathrm{D}^{n+c} + \frac{c}{c-2}\cdot{}^{2}\Delta^{c-2}\cdot{}^{r-2)}\mathrm{D}^{n+c} + \dots$,

$r\cdot{}^{r)}\mathrm{D}^{n+c} - r\,\frac{r-1}{2}\cdot{}^{r)}\mathrm{D}^{n+2c} + \dots + \left(\frac{c}{2}\right)^{r}$,

$-\frac{n}{n+r-1}\cdot{}^{1}\Delta^{n+r-1}\cdot{}^{r-1}\Delta^{1-n-r} - \frac{n}{n+r-2}\cdot{}^{2}\Delta^{n+r-2}\cdot{}^{r-2}\Delta^{2-n-r} - \dots$,

$-\frac{n}{n+r}\left({}^{1)}\mathrm{D}^{-n-r}\cdot{}^{r-1}\Delta^{1-n-r} + {}^{2)}\mathrm{D}^{-n-r}\cdot{}^{r-2}\Delta^{2-n-r} + \dots\right)$,

$n\,\frac{n-1}{n+r}\left(\frac{{}^{r}\Delta^{1-n-r}}{n+r-1} - \frac{{}^{r-1}\Delta^{2-n-r}}{n+r-2} + \frac{{}^{r-2}\Delta^{3-n-r}}{n+r-3} - \dots\right)$,

$\frac{c-n}{c-n-r}\cdot{}^{r}\Delta^{c-n-r} + \frac{c-n}{c-n-r+1}\cdot{}^{1)}\mathrm{D}^{c}\cdot{}^{r-1}\Delta^{c-n-r+1} + \dots$,

$\frac{c-n}{c-n-r}\cdot{}^{r}\Delta^{c-n-r} + \frac{c-n}{c-n-r+1}\,\frac{c}{c+1}\cdot{}^{1}\Delta^{-c-1}\cdot{}^{r-1}\Delta^{c-n-r+1} + \dots$,

$r\,\frac{c-n}{c-n-r}\cdot{}^{r}\Delta^{c-n-r} - r\,\frac{r-1}{2}\,\frac{2c-n}{2c-n-r}\cdot{}^{r}\Delta^{2c-n-r} + \dots + \left(-\frac{c}{2}\right)^{r}$,

$-\frac{n}{n+r}\left({}^{1)}\mathrm{D}^{-n-1}\cdot{}^{r-1}\Delta^{-n-r} + {}^{2)}\mathrm{D}^{-n-2}\cdot{}^{r-2}\Delta^{-n-r} + \dots\right)$,

$-\frac{n+1}{n+r}\cdot{}^{1}\Delta^{n}\cdot{}^{r-1}\Delta^{-n-r} - \frac{n+2}{n+r}\cdot{}^{2}\Delta^{n}\cdot{}^{r-2}\Delta^{-n-r} - \dots$,

$\frac{-n}{2^r-1}\left(\frac{{}^{r-1}\Delta^{1-n-r}}{2} - \frac{n+r+1}{2}\,\frac{{}^{r-2}\Delta^{2-n-r}}{2^2} + \frac{n+r+1}{2}\,\frac{n+r+2}{3}\,\frac{{}^{r-3}\Delta^{3-n-r}}{2^3} - \dots\right)$,

${}^{r}\Delta^{-n-r-1} + {}^{r-1}\Delta^{-n-r-1} + \frac{{}^{r-2}\Delta^{-n-r-1}}{1.2} + \dots$,

$\frac{c-n}{c-n-r}\cdot{}^{r}\Delta^{c-n-r} + \frac{c-n-1}{c-n-r}\cdot{}^{1}\Delta^{-c}\cdot{}^{r-1}\Delta^{c-n-r} + \dots$,

$\frac{c-n}{c-n-r}\cdot{}^{r}\Delta^{c-n-r} + \frac{c-n-1}{c-n-r}\,\frac{c}{c-1}\cdot{}^{1)}\mathrm{D}^{c-1}\cdot{}^{r-1}\Delta^{c-n-r} + \dots.$

Table of ${}^r\Delta^n = \frac{\Delta^n 0^{n+r}}{\Gamma(n+r+1)}$ and of ${}^{r)}\mathrm{D}^n = \frac{\mathrm{D}^n 0^{n+r)}}{\Gamma(n+r+1)}$, accompanying Paragraph 95.

r	${}^r\Delta^{-6}$	${}^r\Delta^{-5}$	${}^r\Delta^{-4}$	${}^r\Delta^{-3}$	${}^r\Delta^{-2}$	${}^r\Delta^{-1}$	${}^r\Delta^{0}$	${}^r\Delta^{1}$	${}^r\Delta^{2}$	${}^r\Delta^{3}$	${}^r\Delta^{4}$	${}^r\Delta^{5}$	${}^r\Delta^{6}$	r
0	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	1
2	$\frac{17}{4}$	$\frac{35}{12}$	$\frac{11}{6}$	1	$\frac{5}{12}$	$\frac{1}{12}$	0	$\frac{1}{6}$	$\frac{7}{12}$	$\frac{5}{4}$	$\frac{13}{6}$	$\frac{10}{3}$	$\frac{19}{4}$	2
3	$-\frac{15}{4}$	$-\frac{25}{12}$	-1	$-\frac{3}{8}$	$-\frac{1}{12}$	0	0	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{3}$	$\frac{25}{8}$	$\frac{21}{4}$	3
4	$\frac{137}{60}$	1	$\frac{251}{720}$	$\frac{19}{240}$	$\frac{1}{240}$	$\frac{-1}{720}$	0	$\frac{1}{120}$	$\frac{31}{360}$	$\frac{43}{120}$	$\frac{81}{80}$	$\frac{331}{144}$	$\frac{1087}{240}$	4
5	-1	$\frac{-95}{288}$	$-\frac{3}{40}$	$\frac{-1}{160}$	$\frac{1}{720}$	0	0	$\frac{1}{720}$	$\frac{1}{40}$	$\frac{23}{160}$	$\frac{37}{72}$	$\frac{45}{32}$	$\frac{259}{80}$	5
6	$\frac{19087}{60480}$	$\frac{863}{12096}$	$\frac{221}{30240}$	$\frac{-1}{945}$	$\frac{-1}{6048}$	$\frac{1}{30240}$	0	$\frac{1}{5040}$	$\frac{127}{20160}$	$\frac{605}{12096}$	$\frac{6821}{30240}$	$\frac{2243}{3024}$	$\frac{30083}{15120}$	6
7	$\frac{-275}{4032}$	$\frac{-95}{12096}$	$\frac{11}{15120}$	$\frac{1}{4032}$	$\frac{-1}{30240}$	0	0	$\frac{1}{40320}$	$\frac{17}{12096}$	$\frac{311}{20160}$	$\frac{265}{3024}$	$\frac{1045}{3024}$	$\frac{97}{90}$	7
8	$\frac{9829}{1209600}$	$\frac{-47}{103680}$	$\frac{-199}{725760}$	$\frac{19}{1209600}$	$\frac{1}{172800}$	$\frac{-1}{1209600}$	0	$\frac{1}{362880}$	$\frac{73}{259200}$	$\frac{2591}{604800}$	$\frac{55591}{1814400}$	$\frac{7501}{51840}$	$\frac{63373}{120960}$	8
9	$\frac{19}{80640}$	$\frac{79}{290304}$	$\frac{1}{604800}$	$\frac{-1}{115200}$	$\frac{1}{1209600}$	0	0	$\frac{1}{3628800}$	$\frac{31}{604800}$	$\frac{437}{403200}$	$\frac{253}{25920}$	$\frac{2669}{48384}$	$\frac{6671}{28800}$	9

r	${}^{r)}\mathrm{D}^{-6}$	${}^{r)}\mathrm{D}^{-5}$	${}^{r)}\mathrm{D}^{-4}$	${}^{r)}\mathrm{D}^{-3}$	${}^{r)}\mathrm{D}^{-2}$	${}^{r)}\mathrm{D}^{-1}$	${}^{r)}\mathrm{D}^{0}$	${}^{r)}\mathrm{D}^{1}$	${}^{r)}\mathrm{D}^{2}$	${}^{r)}\mathrm{D}^{3}$	${}^{r)}\mathrm{D}^{4}$	${}^{r)}\mathrm{D}^{5}$	${}^{r)}\mathrm{D}^{6}$	r
0	1	1	1	1	1	1	1	1	1	1	1	1	1	0
1	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3	1
2	$\frac{13}{4}$	$\frac{25}{12}$	$\frac{7}{6}$	$\frac{1}{2}$	$\frac{1}{12}$	$-\frac{1}{12}$	0	$\frac{1}{3}$	$\frac{11}{12}$	$\frac{7}{4}$	$\frac{17}{6}$	$\frac{25}{6}$	$\frac{23}{4}$	2
3	$\frac{3}{2}$	$\frac{5}{8}$	$\frac{1}{6}$	0	0	$\frac{1}{24}$	0	$-\frac{1}{4}$	$-\frac{5}{6}$	$-\frac{15}{8}$	$-\frac{7}{2}$	$-\frac{35}{6}$	-9	3
4	$\frac{31}{120}$	$\frac{1}{24}$	$\frac{-1}{720}$	$\frac{1}{240}$	$\frac{-1}{240}$	$\frac{-19}{720}$	0	$\frac{1}{5}$	$\frac{137}{180}$	$\frac{29}{15}$	$\frac{967}{240}$	$\frac{1069}{144}$	$\frac{3013}{240}$	4
5	$\frac{1}{120}$	0	0	$\frac{-1}{480}$	$\frac{1}{240}$	$\frac{3}{160}$	0	$-\frac{1}{6}$	$-\frac{7}{10}$	$-\frac{469}{240}$	$-\frac{89}{20}$	$-\frac{285}{32}$	$-\frac{781}{48}$	5
6	$\frac{1}{30240}$	$\frac{-1}{6048}$	$\frac{1}{3024}$	$\frac{1}{945}$	$\frac{-221}{60480}$	$\frac{-863}{60480}$	0	$\frac{1}{7}$	$\frac{363}{560}$	$\frac{29531}{15120}$	$\frac{4523}{945}$	$\frac{81063}{3024}$	$\frac{242537}{12096}$	6
7	0	$\frac{1}{12096}$	$\frac{-1}{3024}$	$\frac{-11}{20160}$	$\frac{19}{6048}$	$\frac{275}{24192}$	0	$-\frac{1}{8}$	$\frac{-761}{1260}$	$-\frac{1303}{672}$	$-\frac{7645}{1512}$	$-\frac{139381}{12096}$	$-\frac{48035}{2016}$	7
8	$\frac{-1}{57600}$	$\frac{-19}{725760}$	$\frac{199}{725760}$	$\frac{47}{172800}$	$\frac{-9829}{3628800}$	$\frac{-33953}{3628800}$	0	$\frac{1}{9}$	$\frac{7129}{12600}$	$\frac{16103}{8400}$	$\frac{341747}{64800}$	$\frac{1148963}{90720}$	$\frac{1666393}{60480}$	8
9	$\frac{1}{57600}$	$\frac{-1}{483840}$	$\frac{-79}{362880}$	$\frac{-19}{161280}$	$\frac{407}{172800}$	$\frac{8183}{1036800}$	0	$-\frac{1}{10}$	$\frac{-671}{1260}$	$-\frac{190553}{100800}$	$-\frac{412009}{75600}$	$-\frac{355277}{25920}$	$-\frac{22463}{720}$	9

96. The factorial theorem may be applied in many cases not contemplated in the foregoing analysis, cases the discussion of which requires the consideration of a class of functions much wider than that already widened class to which I have extended the name of factorial. The chief mark of a factorial, as I have defined it, is the property (266), $\partial x^{[m]} = mx^{[m-1]}$. That this is not the only mark, however, is readily to be seen, since any function of x and m might be made the starting point of a series of functions possessed of that property, to be developed by performing the operation ∂ or ∂^{-1}, it being understood that $m^{-1}\partial\phi(x, m)$ must be called $\phi(x, m-1)$. The form of the function, however, would, in most cases, be liable to perpetual alteration, and it is only such functions as retain their form after being operated upon by ∂ that can to advantage be classed together under the same name.

97. I call $x^{[m]}$ a primary factorial because it is the simplest form of function of which we may say $\partial\phi(x, m) = m\phi(x, m-1)$. If we make $x^{[0]} = 1$, which is clearly its simplest form, we find that

$$x^{[1]} = \partial^{-1}x^{[0]} = x. \tag{347}$$

The possible complementary constant is, for simplicity, disregarded. Similarly,

$$x^{[2]} = 2\partial^{-1}x = x(x + 2ah - h); \tag{348}$$

and by repeated operation we find that (267) is the simplest general form of function to which the property $\partial\phi(x, m) = m\phi(x, m-1)$ can be ascribed.

98. Let Q be any function of E; then, whatever value be assigned to m,

$$\partial \mathrm{Q} x^{[m]} = \mathrm{Q}\partial x^{[m]} = \mathrm{Q} m x^{[m-1]} = m\mathrm{Q}x^{[m-1]}. \tag{349}$$

Let $\mathrm{Q}x^{[m]}$ be represented by $x^{\{m\}}$; then

$$\partial x^{\{m\}} = mx^{\{m-1\}}. \tag{350}$$

In $x^{\{m\}}$ we have, I think, the most general form of function to which the term factorial can conveniently be applied.

99. We can now extend widely the applicability of the factorial theorem. Let $\mathrm{G} = \mathrm{Q}^{-1}$, where Q is any function of E to which that theorem applies. Then, writing $\phi\mathrm{EQ}$ for $\phi\mathrm{E}$ in (291),

$$\phi\mathrm{EG}^{-1} = \phi\mathrm{EQ} = \phi\mathrm{EQ}0^{[0]} + \phi\mathrm{EQ}0^{[1]}.\,\partial + \ldots. \tag{351}$$

Writing $0^{\{m\}}$ for $\mathrm{Q}0^{[m]}$, and multiplying both sides by G, we have the factorial theorem in this more general form,

$$\phi\mathrm{E} = \phi\mathrm{E}0^{\{0\}}.\,\mathrm{G} + \phi\mathrm{E}0^{\{1\}}.\,\mathrm{G}\partial + \frac{1}{2}\,\phi\mathrm{E}0^{\{2\}}.\,\mathrm{G}\partial^2 + \ldots. \tag{352}$$

From this may be drawn deductions similar, *mutatis mutandis*, to those already made from the factorial theorem.

100. Of the forms which G can assume, one of the most important is
$$G = a\mathrm{E}^{-ah} + (1-a)\,\mathrm{E}^{(1-a)h}. \tag{353}$$
In this case, let what $x^{\{m\}}$ becomes be denoted by $x^{]m[}$, employing reversed brackets. Then, for all values of m,
$$x^{]m[} = x^{-1}x^{[m+1]}, \tag{354}$$
as may be shown by performing the operation G on both sides, resulting in the normal equation $\mathrm{G}x^{]m[} = x^{[m]}$. In general, therefore,
$$x^{]m[} = h^m \frac{\Gamma(xh^{-1} + a\,[m+1])}{\Gamma(xh^{-1} + a\,[m+1] - m)}, \tag{355}$$
and when m is positive and integral,
$$x^{]m[} = (x + a\,[m+1]\,h - h)(x + a\,[m+1]\,h - 2h) \ldots (x + [a-1][m+1]\,h + h). \tag{356}$$
Let us call all functions of this form Secondary Factorials. Just as we have distinguished certain functions as primary factorials, because they are the simplest functions complying with the conditions $\partial\phi\,(x, m) = m\phi(x, m-1)$ and $\phi\,(x, 0) = x^0$, so we may remark in this case that the class of secondary factorials comprises those which comply with the conditions $\partial\phi\,(x, m) = m\phi\,(x, m-1)$ and $\phi\,(x, -1) = x^{-1}$.

101. The investigation of negative factorials is not required in connection with the object for which I have embraced the theory of factorials in this essay, namely, the development and illustration of the factorial theorem. I shall, therefore, dismiss that branch of the subject with but transient consideration. When m is a negative integer, we have, as the general form of primary factorials,
$$x^{[m]} = \frac{x}{(x + amh)(x + amh - h) \ldots (x + [a-1]\,mh)}; \tag{357}$$
and, when m is any negative integer (except -1, when $x^{]-1[} = x^{-1}$),
$$x^{]m[} = \frac{1}{(x + a\,[m+1]\,h)(x + a\,[m+1]\,h - h) \ldots (x + [a-1]\,[m+1]\,h)} \tag{358}$$
as the general form of secondary factorials. By repeated operation,
$$\partial^n x^{]-1['} = \partial^n x^{]-1[} = (-1)^n\,\Gamma(1+n)\,x^{]-n-1[}. \tag{359}$$
But, by (338),
$$\begin{aligned}\partial^n x^{]-1['} &= (\partial'^n + {}^{[1]'}\partial^n \,.\, \partial'^{n+1} + \ldots)\,x^{]-1['}\\ &= (-1)^n\,\Gamma(1+n)\,(x^{]-n-1['} - [n+1]\,.{}^{[1]'}\partial^n \,.\, x^{]-n-2['} + \ldots)\,.\end{aligned} \tag{360}$$
Comparing these two expressions, and writing $n-1$ for n, we have this general theorem for the transformation of one form of negative integral secondary factorial into another,
$$x^{]-n[} = x^{]-n['} - n\,.{}^{[1]'}\partial^{n-1} \,.\, x^{]-n-1['} + n\,(n+1)\,.{}^{[2]'}\partial^{n-1} \,.\, x^{]-n-2['} - \ldots. \tag{361}$$

Since $x^{[-n]} = xx^{]-n-1[}$, we obtain also the following general theorem for the transformation of negative integral primary factorials,

$$x^{[-n]} = x^{[-n]\prime} - (n+1)\,.^{[1]\prime}\partial^{n}.\,x^{[-n-1]\prime} + (n+1)(n+2)\,.^{[2]\prime}\partial^{n}.\,x^{[-n-2]\prime} - \ldots . \quad (362)$$

102. We may distinguish the principal special cases of secondary, and also negative, factorials in a manner similar to that employed for the distinction of the corresponding special cases of positive primary factorials. The following table, showing the chief special forms of $x^{[3]}$ and $x^{[-3]}$, $x^{]3[}$ and $x^{]-3[}$, will afford a sufficient illustration of the use of the various special symbols which represent the most important varieties of factorials.

		h	a	Primary.	Secondary.
Positive:	Power	0		$x^{3} = xxx$	$x^{3} = xxx$
	Upper	1	0	$x^{3)} = x(x-1)(x-2)$	$x^{3(} = (x-1)(x-2)(x-3)$
	Lower	1	1	$x^{(3} = x(x+2)(x+1)$	$x^{)3} = (x+3)(x+2)(x+1)$
	Central	1	$\frac{1}{2}$	$x^{(3)} = x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$	$x^{)3(} = (x+1)\,x\,(x-1)$
Negative:	Power	0	·	$x^{-3} = \frac{1}{xxx}$	$x^{-3} = \frac{1}{xxx}$
	Upper	1	0	$x^{-3)} = \frac{1}{(x+1)(x+2)(x+3)}$	$x^{-3(} = \frac{1}{x(x+1)(x+2)}$
	Lower	1	1	$x^{(-3} = \frac{1}{(x-3)(x-2)(x-1)}$	$x^{)-3} = \frac{1}{(x-2)(x-1)\,x}$
	Central	1	$\frac{1}{2}$	$x^{(-3)} = \frac{x}{\left(x-\frac{3}{2}\right)\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\right)}$	$x^{)-3(} = \frac{1}{(x-1)\,x\,(x+1)}$

Exclusive of x^{m}, the most important forms of $x^{[m]}$ are $x^{m)}$, $x^{(m)}$, and $x^{)m(}$, corresponding respectively to the three most important forms of difference, Δ, Λ and $\mathrm{I}\Lambda$. Lower differences and lower factorials are comparatively unimportant, since $\phi\,{}^{\backprime}\Delta\psi x = \phi(-\Delta)\,\psi(-y)$, where $y = -x$, a slight transformation thus enabling us to use Δ instead of ${}^{\backprime}\Delta$.

103. For expansion in terms of mean central differences, we may, in (352), write I for G and $0^{)m(}$ for $0^{\{m\}}$, when the factorial theorem assumes this shape:

$$\phi\mathrm{E} = \phi\mathrm{E}0^{)0(}.\,\mathrm{I} + \phi\mathrm{E}0^{)1(}.\,\mathrm{I}\Lambda + \frac{1}{2}\,\phi\mathrm{E}0^{)2(}.\,\mathrm{I}\Lambda^{2} + \ldots . \quad (363)$$

To illustrate this formula, let $\phi\mathrm{E} = \mathrm{E}^{0}$, and we have at once the well known formula for interpolation, due to Gauss, and now assuming the following symmetrical form:

$$\mathrm{E}^{0} = \mathrm{I} + \frac{0^{)2(}}{2}\,\mathrm{I}\Lambda^{2} + \frac{0^{)4(}}{2.3.4}\,\mathrm{I}\Lambda^{4} + \ldots . \quad (364)$$

Again, let $\phi\mathrm{E} = \mathrm{D}$; then

$$\mathrm{D} = \mathrm{I}\Lambda + \frac{\mathrm{D}0^{)3(}}{2.3}\,\mathrm{I}\Lambda^{3} + \frac{\mathrm{D}0^{)5(}}{2.3.4.5}\,\mathrm{I}\Lambda^{5} + \ldots . \quad (365)$$

Thus is disclosed, in the neatest possible form, the law of the known series (256). Another special case of (363) is the following important, and probably new, summation formula:

$$\frac{1}{1+\mathrm{E}} = \frac{1}{2} - \frac{1}{4}\,\mathrm{I}\Delta + \frac{1}{4^2}\,\mathrm{I}\Delta^3 - \frac{1}{4^3}\,\mathrm{I}\Delta^5 + \ldots . \tag{366}$$

This formula, which is otherwise easily demonstrable, since $\dfrac{1}{1+\mathrm{E}} = \dfrac{1}{2} - \dfrac{\mathrm{I}\Delta}{4+\Delta^2}$, may be interpreted as follows. If $\ldots a_{-1},\ a_0,\ a_1, \ldots$ be any series,

$$a_0 - a_1 + a_2 - \ldots = \frac{1}{2}\,a_0 - \frac{1}{4}\,\mathrm{I}\Delta a_0 + \frac{1}{4^2}\,\mathrm{I}\Delta^3 a_0 - \ldots . \tag{367}$$

This theorem will doubtless be found at least equal in usefulness to the corresponding formula in upper differences, and superior to Hutton's method of summing quantities alternately positive and negative. It is almost needless to say that $\mathrm{I}\Delta^n$ may be written in (332) for ∂^m, or, if $0^{)m(}$ is also written for $0^{[m]}$, for ∂^n, according as it is desired to transform mean central differences, positive or negative, into other forms of difference-ratio, or *vice versa.* It does not seem necessary, for present purposes, to enter into a recital of the proof of this statement. The coefficients represented by ${}^r\mathrm{I}\Delta^n$ and ${}^{)r(}\mathrm{D}^n$ are of considerable importance, and should be tabulated.

VII. *Theory of the Calculus of Multiplication.*

104. The Calculus of Enlargement is, as we have seen, based on that operation which changes ϕx into $\phi\,(x+h)$ by adding to the variable. If we seek the most simple repetitive operation which shall have the effect of multiplying x, instead of adding to it, we shall find that it consists in changing ϕx into $\phi\,(x\varepsilon^h)$. Let us denote this operation, as in paragraph 34, by the symbol M, so that

$$\mathrm{M}^h\phi x = \phi\,(x\varepsilon^h)\,. \tag{368}$$

The operation E^h, the basis of the Calculus of Enlargement, changes x in arithmetical ratio, so to speak, while the operation M^h, the basis of what we may call the Calculus of Multiplication, changes it in geometrical ratio.

105. We have seen that in this case $\mathrm{M} = \mathrm{S}$ where $\psi x = \log x$, and that all the results derivable from a possible Calculus of Multiplication can be obtained at once from the Calculus of Enlargement by expressing functions of x as functions of $u = \log x$, and observing that $\mathrm{M}_x = \mathrm{E}_u$. It is, therefore, unnecessary for any practical purpose to discuss that possible calculus.

Nevertheless such a discussion will not now be useless, for it will serve to illustrate and impress upon the mind the truth that the Calculus of Enlargement is not the only possible calculus, but is rather to be regarded as the simplest of many possible systems.

106. Let log M be represented by L; then, by (69),

$$\mathrm{M}^h = 1 + h\mathrm{L} + \frac{1}{2} h^2\mathrm{L}^2 + \ldots, \tag{369}$$

$$\phi(x\varepsilon^h) = \phi x + h\mathrm{L}\phi x + \frac{1}{2} h^2\mathrm{L}^2\phi x + \ldots. \tag{370}$$

Putting $x = 1$, and afterwards writing ϕ log for ϕ, we have

$$\phi(\varepsilon^h) = \phi 1 + h\mathrm{L}\phi 1 + \frac{1}{2} h^2\mathrm{L}^2\phi 1 + \ldots, \tag{371}$$

$$\phi h = \phi 0 + h\mathrm{L}\phi \log 1 + \frac{1}{2} h^2\mathrm{L}^2\phi \log 1 + \ldots. \tag{372}$$

These three theorems are respectively analogous to those of Taylor, Herschel, and Maclaurin. Assuming, what will be proved, that

$$\phi\mathrm{L}\psi \log 1 = \psi\mathrm{L}\phi \log 1, \tag{373}$$

we have also

$$\phi(\varepsilon^h) = \phi 1 + h\phi\mathrm{M} \log 1 + \frac{1}{2} h^2\phi\mathrm{M}(\log 1)^2 + \ldots, \tag{374}$$

$$\phi h = \phi 0 + h\phi\mathrm{L} \log 1 + \frac{1}{2} h^2\phi\mathrm{L}(\log 1)^2 + \ldots. \tag{375}$$

107. From (75),

$$\mathrm{L} \log x = 1. \tag{376}$$

From (77) and (78),

$$\phi\mathrm{M}_x\psi(u, v, w, \ldots) = \phi(\mathrm{M}_{x\,|\,u}\mathrm{M}_{x\,|\,v}\mathrm{M}_{x\,|\,w}\ldots)\,\psi(u, v, w, \ldots), \tag{377}$$

$$\phi\mathrm{M}_{x\,|\,u}\psi(u, v) = \phi(\mathrm{M}_x\mathrm{M}_{x\,|\,v}{}^{-1})\,\psi(u, v). \tag{378}$$

From (88),

$$\mathrm{L}_x v = \mathrm{L}_x \log u \,.\, \mathrm{L}_u v. \tag{379}$$

108. By comparison of the general terms of the expansions of the two members in each case, respectively, we find that

$$\phi\mathrm{M}_x\psi(xy) = \phi\mathrm{M}_y\psi(xy), \tag{380}$$

$$\phi(y\mathrm{M}_x)\,\psi x = \psi(x\mathrm{M}_y)\,\phi y, \tag{381}$$

$$\phi\mathrm{M}x^m\psi x = x^m\phi(\varepsilon^m\mathrm{M})\,\psi x, \tag{382}$$

$$\phi(\mathrm{M}_x\mathrm{M}_y\ldots)\,s = s\phi(\varepsilon^n), \tag{383}$$

where s is a quantic of the nth degree, say $s = x^hy^k\ldots$, where $h + k + \ldots = n$; also

$$\phi(\mathrm{M}_x\mathrm{M}_y\ldots)\,s\psi(x, y, \ldots) = s\phi(\varepsilon^n\mathrm{M}_x\mathrm{M}_y\ldots)\,\psi(x, y, \ldots). \tag{384}$$

If, in (381), $y = 1$,

$$\phi\mathrm{M}\psi x = \psi(x\mathrm{M}_1)\,\phi 1, \tag{385}$$

$$\phi \text{M} x^m = x^m \text{M}_1^m \phi 1 = x^m \phi(\varepsilon^m)\,, \tag{386}$$

$$\phi \text{M}_x c = c\phi 1\,, \tag{387}$$

c being anything independent of x.

109. Writing, in the foregoing formulæ, ϕ log for ϕ, we have these general theorems,

$$\phi \text{L}_x \psi(u, v, w, \ldots) = \phi(\text{L}_{x\,|\,u} + \text{L}_{x\,|\,v} + \text{L}_{x\,|\,w} + \ldots)\,\psi(u, v, w, \ldots), \tag{388}$$

$$\phi \text{L}_{x\,|\,u} \psi(u, v) = \phi(\text{L}_x - \text{L}_{x\,|\,v})\,\psi(u, v), \tag{389}$$

$$\phi \text{L}_x \psi(xy) = \phi \text{L}_y \psi(xy), \tag{390}$$

$$\phi(\log y + \text{L}_x)\,\psi x = \psi(x\text{M}_y)\,\phi \log y, \tag{391}$$

$$\phi \text{L} x^m \psi x = x^m \phi(m + \text{L})\,\psi x, \tag{392}$$

$$\phi(\text{L}_x + \text{L}_y + \ldots)\,s = s\phi n, \tag{393}$$

$$\phi(\text{L}_x + \text{L}_y + \ldots)\,s\psi(x, y, \ldots) = s\phi(n + \text{L}_x + \text{L}_y + \ldots)\,\psi(x, y, \ldots), \tag{394}$$

$$\phi \text{L} \psi x = \psi(x\text{M}_1)\,\phi \log 1, \tag{395}$$

$$\phi \text{L} x^m = x^m \phi m, \tag{396}$$

$$\phi \text{L} c = c\phi 0. \tag{397}$$

From (395), writing ψ log for ψ, and putting $x = 1$, we have (373). As special cases of some of these general theorems, we obtain

$$\text{L}_x^n \psi(u, v, \ldots) = (\text{L}_{x\,|\,u} + \text{L}_{x\,|\,v} + \ldots)^n \psi(u, v, \ldots), \tag{398}$$

$$\text{L}_x uv = u\text{L}_x v + v\text{L}_x u\,, \tag{399}$$

$$\text{L}_{x\,|\,u}{}^n \psi(u, v) = (\text{L}_x - \text{L}_{x\,|\,v})^n \psi(u, v)\,, \tag{400}$$

$$\text{L}^n x^m = x^m m^n, \tag{401}$$

$$\text{L} x^m = m x^m, \tag{402}$$

$$\text{L} c = 0\,. \tag{403}$$

110. From (379), since $\text{L}_u u = u$,

$$\text{L}_x u = u\text{L}_x \log u\,, \tag{404}$$

$$\text{L}\varepsilon^{\phi x} = \varepsilon^{\phi x}\text{L}\phi x\,, \tag{405}$$

$$\text{L}\varepsilon^{kx} = \varepsilon^{kx}\text{L}kx = \varepsilon^{kx}kx\,. \tag{406}$$

To illustrate (404),

$$\text{L} x^m = x^m \text{L} m \log x = m x^m, \tag{407}$$

$$\text{L}_x \log u = \frac{\text{L}_x u}{u}, \tag{408}$$

$$\text{L} \log x = \frac{\text{L}x}{x} = 1\,. \tag{409}$$

111. If we know the development of $\phi(x\varepsilon^h)$ in positive integral powers of h, we can tell from it the value of $\text{L}\phi x$, which, by (370), is the coefficient of h in such development. Thus, since $x^m \varepsilon^{hm} = x^m(1 + hm + \ldots)$, we have $\text{L}x^m = mx^m$, and since $\log(x\varepsilon^h) = \log x + h$, we have $\text{L} \log x = 1$. Again, the

various special cases of the general logarithmic series (31) will afford equivalents for L in terms of M or of simple functions of M, by means of which we may ascertain $\text{L}\phi x$. It is needless to recite these cases. As illustrations,

$$\text{L}x^m = \frac{\text{M}^0 - 1}{0} x^m = x^m \frac{\varepsilon^{m0} - 1}{0} = mx^m, \tag{410}$$

$$\text{L} \log x = \frac{\log (x\varepsilon^0) - \log x}{0} = 1, \tag{411}$$

$$\text{L}x^m = (\text{M} - 1)\, x^m - \frac{1}{2} (\text{M} - 1)^2 x^m + \ldots = x^m([\varepsilon^m - 1] - \ldots) = mx^m. \tag{412}$$

112. It follows from (403) that, when the inverse operation L^{-1} is performed, a complementary constant must be introduced. Performing that operation on (399) we have

$$\text{L}_x^{-1} u \text{L}_x v = uv - \text{L}_x^{-1} v \text{L}_x u\,. \tag{413}$$

For example,

$$\begin{aligned} \text{L}^{-1} x\varepsilon^x &= x\varepsilon^x - \text{L}^{-1} x^2 \varepsilon^x \\ &= x\varepsilon^x - \frac{1}{2} x^2 \varepsilon^x + \text{L}^{-1} \frac{1}{2} x^3 \varepsilon^x \\ &= x\varepsilon^x - \frac{1}{2} x^2 \varepsilon^x + \frac{1}{2\,.\,3} x^3 \varepsilon^x - \ldots + c\,. \end{aligned} \tag{414}$$

But $\text{L}^{-1} x\varepsilon^x = \varepsilon^x$. Substituting this in (414), dividing throughout by ε^x, and writing $-x$ for x, we have

$$c\varepsilon^x = 1 + x + \frac{1}{2} x^2 + \ldots, \tag{415}$$

wherein putting $x = 0$ shows that $c = 1$.

113. If ϕx is algebraically less than $\phi\,(x\varepsilon^h)$ for all values of h lying between some positive quantity and some other negative quantity, exclusive of the value $h = 0$, it is a minimum, and if greater than $\phi\,(x\varepsilon^h)$ for all such values, a maximum. If, in

$$\phi\,(x\varepsilon^h) = \phi x + h\text{L}\phi x + \frac{1}{2} h^2 \text{L}^2 \phi x + \ldots, \tag{416}$$

$\text{L}\phi x$ does not vanish, ϕx is neither a maximum nor a minimum, unless, indeed, $\text{L}\phi x$ is infinite, a case which we need not now consider; for, by making h small enough to cause $h\text{L}\phi x$ to exceed the sum of all succeeding terms, $\phi\,(x\varepsilon^h) - \phi x$ and $\phi\,(x\varepsilon^{-h}) - \phi x$ will have different signs. If $h\text{L}\phi x = 0$,

$$\phi\,(x\varepsilon^h) = \phi x + \frac{1}{2} h^2 \text{L}^2 \phi x + \ldots, \tag{417}$$

and when h is small enough to cause $\frac{1}{2} h^2 \text{L}^2 \phi x$ to exceed the sum of all succeeding terms, $\phi\,(x\varepsilon^h) - \phi x$ and $\phi\,(x\varepsilon^{-h}) - \phi x$ will have the same sign, and ϕx

will be a maximum or a minimum, provided $\text{L}^2\phi x$ does not vanish, in which case the matter is still in doubt. If $\text{L}^2\phi x$ is negative, ϕx is a maximum, and *vice versa.* It may in this manner be shown that for ϕx to be a maximum or minimum an odd number of powers of h must vanish, in which case the coefficient of the next succeeding power will, by its sign, determine whether ϕx is a maximum or a minimum. For example, let $\phi x = \varepsilon^{1-x} + \varepsilon^{x-1} + 2\cos(x-1)$. In this case it will be found that, when $x = 1$, $\text{L}\phi x$, $\text{L}^2\phi x$ and $\text{L}^3\phi x$ all vanish, and $\text{L}^4\phi x = 4$, showing that ϕx is a minimum.

114. Since the processes of any calculus may be expressed in the language of any other calculus, those of the Calculus of Enlargement may be expressed in the language of the Calculus of Multiplication. For example, let us in (370) write ϕ log for ϕ and x for log x; then, if $u = \varepsilon^x$,

$$\phi(x+h) = \phi x + h\text{L}_u\phi x + \frac{1}{2}h^2\text{L}_u^2\phi x + \dots, \tag{418}$$

a form of Taylor's theorem. Since by (379) $\text{L}_u = x^{-1}\text{L}_x$,

$$\phi(x+h) = \phi x + hx^{-1}\text{L}_x\phi x + \frac{1}{2}h^2(x^{-1}\text{L}_x)^2\phi x + \dots. \tag{419}$$

The results of the Calculus of Enlargement may thus, in general, be obtained by the processes of the Calculus of Multiplication; though the former method must, of course, be preferred on the ground of simplicity. For expressing the results of the Calculus of Multiplication in the language of the Calculus of Enlargement we have, from (93),

$$\text{L}_x = \text{L}_x x \,.\, \text{D}_x = x\text{D}_x\,. \tag{420}$$

C. Summary.

115. The Calculus of Enlargement relates to the theory and practice of certain operations, embracing in its field the Calculus of Finite Differences, the Differential and Integral Calculus, and the Calculus of Variations. The operations comprised by it are those whose symbols are functions of E, the symbol of Enlargement, the operation by which ϕx becomes $\phi(x+1)$. It would be possible to form any number of systems, each a calculus comprising operations whose symbols are functions of some simple symbol other than E, but the results so obtainable can be got from the Calculus of Enlargement, and the elaboration of such other systems would, therefore, be superfluous.

116. All functions of E may be treated separately from the subject of operation, by any algebraic rules applicable to symbols in general, the theory

of the functions of E forming an Algebra of which the theory of Differentiation is that part corresponding to the theory of Logarithms in ordinary Algebra. The symbol of Differentiation, D, is the logarithm of E, the symbol of Enlargement. Whatever theorems may be proved regarding functions of E, as such, are true of D as one such function. It is worth remarking that at first the theory of logarithms was treated in a far-fetched and comparatively obscure manner, in connection with the properties of the hyperbola; many years passing before it was reduced to a simpler form as a branch of algebra. It is not without historical analogy, therefore, that the doctrine of differentiation has hitherto failed to find its true place as the logarithmic branch of that wider algebra, the doctrine of the functions of E, or Calculus of Enlargement.*

117. For the direct interpretation of D it is necessary to write E for x in expressions giving log x in terms of x or of simple functions of x. The two general logarithmic theorems (30) and (31) embrace, as special cases, many such expressions. For the better understanding of logarithms, it is best to refrain, in the definition of a logarithm, from describing it as an exponent. To explain differentials, we have the definition $d = \log e$, where e represents enlargement with respect to a hypothetical variable; and also, for infinitesimal differentials, the equation $d = \mathrm{E}^{(1-\alpha)0} - \mathrm{E}^{-\alpha 0}$. The most important expressions equivalent to D are three principal vanishing fractions and three corresponding series. The symbolic vanishing fractions appear to be novel, though the practical application of one or two of them is familiar; while, on the other hand, the series are well known in their symbolic form, though their practical use as definitions of D does not seem to have been previously suggested. These fractions and series are special cases of a single differentiate-expression (220), which is itself a special case of the factorial theorem. Taken all together, and in connection with other equivalents of D, they convey a broad and comprehensive idea of the meaning of the operation of differentiation; the notion afforded by the vanishing fractions being at once the hardest to grasp and the most satisfactory when clearly understood.

*"In fact, the arrangement of the truths of analytical science, such as history gives it, is very different from their logical and natural arrangement; and as, in the infancy of analysis, mathematicians were more solicitous to advance it, than to advance it by just and natural means, they frequently deviated into indirect and foreign demonstrations. . . . The evil attending on this mode of procedure has been, . . . that the principles of a general method have been sought for in some particular method, properly (that is, according to the logical and natural order of ideas) to be comprehended under the general one." Woodhouse, *Analytical Calculations*, p. 40. This was written relative to the history of log x, but applies equally well to that of log E.